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THE SPECTRUM OF FREQUENCY-MODULATED WAVES AFTER RECEPTION IN RANDOM NOISE-I*

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Part I: Introduction and Discussion

The reception of radio, radar, and television signals is always accomplished in the presence of noise, which exclusive of outside sources—either man-made or natural—arises because of the random motion of electrons in the various circuit elements of the receiver, principally in the stages preceding the detecting devices. The noise is accordingly assumed to be of the fluctuation type; impulse noise is not considered in this paper. Knowledge of the spectral distribution and the power output following demodulation of a frequency-modulated carrier in such noise is essential to any critical theory of receiver behavior. Accordingly, the principal purpose of the present work is to investigate the spectral aspects of the problem. One of the chief reasons for our interest in the spectrum lies in the fact that the signal-to-noise ratio at the output as a function of the ratio at the input depends markedly on the spectral distribution, in the so-called case of *broad-band FM* (where the maximum change in carrier frequency may be several times the highest modulating components). The dependence in the instance of *narrow-band FM* (for which the maximum deviation in the carrier frequency is comparable with or less than the highest significant audio modulating frequency) is not nearly so critical.

Previous investigations, of which the work of Carson and Fry,[1] Crosby,[2] Blachman,[3] and Rice[4] are particularly to be noted here,** have dealt chiefly with the strong-carrier case, in which the noise is largely suppressed and appears linearly with the signal in

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[1] J. R. Carson and T. C. Fry, *Variable frequency electric circuit theory with application to the theory of frequency modulation*, Bell Syst. Tech. J. **16**, 513 (1937).

[2] M. C. Crosby, *Frequency modulation noise characteristics*, Proc. I. R. E. **25**, 472 (1937).

[3] N. M. Blachman, *A theoretical study of the demodulation of a frequency-modulated carrier and random noise by an FM receiver*, Thesis (Harvard, June, 1947). Technical Report No. 31, (1948), Cruft Laboratory, Harvard University. See also N. M. Blackman, J. Appl. Phys. **20**, 38 (1949).

[4] S. O. Rice, *Statistical properties of a sine-wave and random noise*, Bell Syst. Tech. J. **27**, 109 (1948).

**An interesting paper by F. L. H. M. Stumpers (Proc. I.R.E. 36, 1080 (1948) has since appeared, giving some of the same general conclusions reached here in the case of extreme limiting. The case of no limiting, however, is not discussed.

the output. Rice's work[4] is an exception, in that he determines the spectrum for the case of arbitrary input carrier-to-noise power ratio (p), but only for the idealized example of extreme limiting (as do likewise Carson and Fry[1] and Crosby[2]). Further, the carrier is not modulated. Blachman's interesting treatment,[3] on the other hand, is more comprehensive, since he includes modulation and a discussion of the case of no limiting. Similarly, the present paper is more general than Rice's work in that modulation and no limiting are also considered, and is an extension of Blachman's efforts[5] since spectra (and their associated correlation functions) are obtained for all values of the ratio p.

We consider the case of no limiting because it gives us one extreme of receiver performance as a function of limiting; superlimiting gives us the other. Since the passage from the former to the latter is monotonic, we may be able roughly to estimate the behavior for intermediate degrees of limiting. The ability to handle all values of input carrier-to-noise power ratios is particularly useful when we desire the spectrum for threshold signals ($p \backsim 1$ or less), an important region when contrasting the maximum sensitivities of AM and FM reception. Specific calculations of spectra are made, which utilize a Gaussian input noise distribution. This spectral shape is usually a somewhat better fit to actual responses than the rectangular spectrum. The results of the analysis are given in detail in Part II, while Part III contains a treatment of important special cases: (1) noise alone, (2) unmodulated carrier and noise, and (3) a sinusoidally modulated carrier in noise.

The actual nonlinear or demodulating elements in an FM receiver are too involved to yield satisfactorily to direct treatment, but a possible model of these essential circuit elements—the limiter, followed by the discriminator—may be constructed if we assume that the physical discriminator is replaced by an "ideal" one which responds everywhere linearly with frequency (the "quasi-stationary" hypothesis of Carson and Fry). Thus, the output current (or voltage) is directly proportional to the instantaneous difference frequency between the wave and the central or resonant frequency of the IF, limiter, and discriminator bands (all tuned to the same frequency and assumed to be symmetrical). The filter characteristic of the limiter is taken to be wide enough to pass the IF portion of the limited signal and noise without distortion due to frequency selection. In practice, of course, the real discriminator saturates for sufficiently large excursions about resonance, but such behavior is rare if only an insignificant portion of the wave's energy is redistributed into higher harmonics due to the "spreading" action of the limiter.[6] This is assumed to be true here even for the extremes of superlimiting, in which the incoming disturbance is heavily clipped at top and bottom.

[5]Blachman has shown (ref. 3, ch. IV) that when there is no limiting, the characteristic function method used by Rice and others[8] in the solution of AM receiver problems may be adapted for all carrier strengths to the problem in which the idealized discriminator is replaced by one closer to actual electronic practice, namely, two diode rectifiers π radians out of phase with one another. Unfortunately, the method breaks down when there is limiting, since it does not seem practicable to determine, in any useful form, the distribution functions W_1 and W_2 of the clipped, *filtered* noise and signal waves leaving the limiter to enter a second nonlinear device of the above type. Kac and Siegert (J. Appl. Phys. 18, 383 (1947)) have calculated the first-order distribution W_1 , but it is the second-order density W_2 that is needed for the spectrum. (The analysis of the present paper has been extended by the author to include arbitrary limiting, with the idealized discriminator and similar assumptions as to the width of the limiter's filter response. See Technical Report No. 62, Cruft Laboratory, Harvard University, Nov. 24, 1948).

[6]D. Middleton, *The response of biased, saturated linear and quadratic rectifiers to random noise*, J. Appl. Phys. **17**, 778 (1946).

The wave leaving the *IF* and entering the limiter is narrow-band, that is to say, it may be represented analytically by

$$V(t) = R(t) \cos [\omega_0 t + \theta(t)], \tag{1.1}$$

where $f(=\omega_0/2\pi)$ is the central frequency of the *IF*, limiter, and discriminator, R is the envelope and θ a phase angle; both R and θ are slowly varying functions of time compared with $\omega_0 t$. If we let $f(iz)$ be the Fourier transform of the limiter's dynamic characteristic $g(V)$, i.e.

$$f(iz) = \int_0^\infty e^{-izV} g(V)\, dV, \tag{1.2}$$

the output of the limiter may be written[7,8]

$$V_0(t) = \frac{1}{2\pi} \int_{\mathbf{C}} f(iz) \exp [iz R \cos (\omega_0 t + \theta)]\, dz. \tag{1.3}$$

The contour **C** extends along the real axis from $-\infty$ to $+\infty$ and is indented downward about a possible singularity at the origin. With the help of the expansion

$$\exp (ia \cos \phi) = \sum_{n=0}^{\infty} i^n \epsilon_n J_n(a) \cos n\phi, \qquad \epsilon_0 = 1, \qquad \epsilon_{n\geq 1} = 2, \tag{1.4}$$

we obtain from (1.3)

$$V_0(t) = \sum_{n=0}^{\infty} B_n(R) \cos n(\omega_0 t + \theta), \tag{1.5}$$

where

$$B_n(R) = \frac{i^n \epsilon_n}{2\pi} \int_{\mathbf{C}} f(iz) J_n(Rz)\, dz. \tag{1.6}$$

The quantities $B_n(R)$ are the *envelopes* of the n $(=0, 1, 2, \cdots)$ spectral zones that are produced in the limiter by its nonlinear action;[7,8] $n\theta$ is the phase relative to $n\omega_0 t$, the n-th harmonic of the original central (angular) frequency. Only the band centered about $f_0 (n = 1)$ is passed to enter the discriminator. The input wave is therefore

$$V_{i-d} = B_1(R) \cos (\omega_0 t + \theta). \tag{1.7}$$

In vector language, this input is represented by $\mathbf{V}_{i-d} = V\mathbf{e}^{i\Phi}$, where real quantities correspond to the x-components, and imaginary quantities to the y-components. Here $V = B_1(R)$ and $\Phi = \omega_0 t + \theta$. Then we have

$$\dot{V}_{i-d} = \dot{V}(\cos \Phi + i \sin \Phi) + V\dot{\Phi}(-\sin \Phi + i \cos \Phi) = \mathbf{r}\dot{V} + \boldsymbol{\theta} V\dot{\Phi},$$

since the polar unit vectors $(\mathbf{r}, \boldsymbol{\theta})$ are related to the rectangular ones by the transformation

$$\mathbf{r} = \cos \Phi + i \sin \Phi, \qquad \boldsymbol{\theta} = -\sin \Phi + i \cos \Phi;$$

$\dot{V}$ is the time rate-of-change of the modulus, and $V\dot{\Phi}$ is the time-derivative of the

[7] S. O. Rice, *Mathematical analysis of random noise*, Bell Syst. Tech. J. **24**, 46 (1945).

[8] D. Middleton, *Some general results in the theory of noise through nonlinear devices*, Quart. Appl. Math. **5**, 445 (1948). (D. M. I.)

phase, or, what is the same thing, the instantaneous frequency.[9] The output E_0' of our idealized discriminator is precisely this second term, namely

$$E_0'(t) = \kappa V\dot{\Phi} = \kappa B_1(R)(\omega_0 + \dot{\theta}),$$

where κ is a constant of proportionality with the dimensions (secs). In practice the high frequency term, represented above as the coefficient of ω_0 , is not passed by the discriminator filter, so that the final discriminator output is

$$E_0(t) = \kappa B_1(R)\dot{\theta} = \frac{i\dot{\theta}\kappa}{\pi}\int_{\mathbf{C}} f(iz)J_1(Rz)\,dz. \tag{1.9}$$

Notice that the effect of the limiter is contained in the term $B_1(R)$, the exact form of which depends on our choice of $g(V)$, cf. (1.2). We assume that the limiter amplifies linearly (for both positive and negative amplitudes) up to a certain level, denoted by

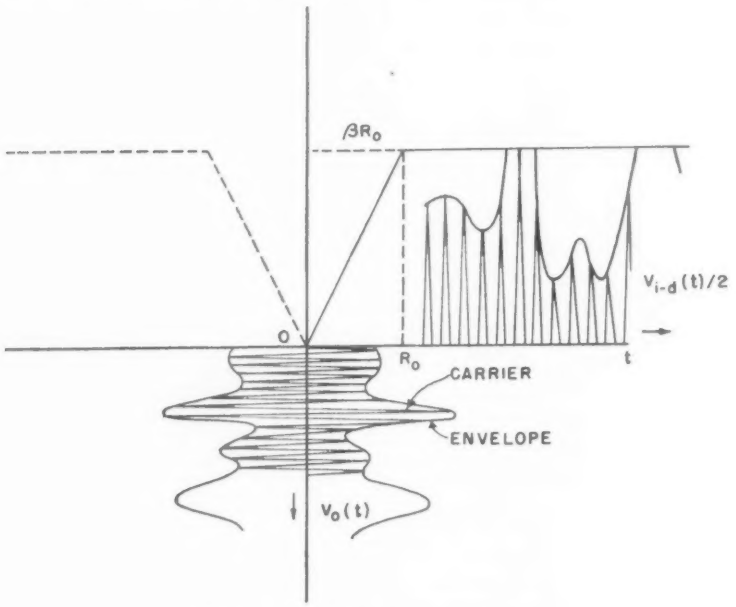

FIG. 1. A typical limiter characteristic.

R_0 , beyond which point saturation occurs abruptly. The dynamic characteristic is shown schematically in Fig. 1. From (1.2) we find that here

$$f(iz) = 2\beta(1 - \exp[-iR_0z])/(iz)^2, \tag{1.10}$$

where β is the dynamic transfer constant of the limiter and the factor 2 is present be-

[9] B. van der Pol, *The fundamental principles of frequency modulation*, Jour. I. E. E. 93, Part III, 153 (1946).

cause both the positive and negative portions contribute to the *envelope*. Accordingly, $B_1(R)$ in (1.9) is[10]

$$B_1(R) = -\frac{2\beta i}{\pi}\int_C (1 - e^{-iR_0 z})J_1(Rz)\frac{dz}{z^2} = \beta R, \qquad 0 \le R \le R_0\,, \tag{1.11a}$$

$$\left.\begin{aligned} &= \frac{4}{\pi}\beta R_0\, {}_2F_1(-1/2,\ 1/2;\ 3/2;\ R_0^2/R^2), \\ &= \frac{2}{\pi}\beta R\left\{\frac{R_0}{R}\left(1 - \frac{R_0^2}{R^2}\right)^{1/2} + \sin^{-1}\frac{R_0}{R}\right\} \end{aligned}\right\} \quad R_0 \le R. \tag{1.11b}$$

The two cases of chief interest are (a), *no limiting*, and (b), *superlimiting*. From (1.11a) and (1.11b) it is evident that $B_1(R) = \beta R$ (no limiting); and $B_1(R) \doteq (4/\pi)\beta R_0$ (superlimiting) in which the *rms* saturation level for the envelope is much less than the input noise level $(b_0)^{1/2}$, i.e. $R_0^2 \ll 2b_0$.

Although the spectral behavior of the discriminator output depends on a considerable number of parameters, chief among them (1) *carrier strength*, (2) *degree of limiting*, (3) *power and spectral distribution of the noise entering the demodulating elements*, (4) *deviation from resonance*, (5) *intensity and wave-form of the modulation*, etc., certain general observations can be made with regard to these factors and their effect on the power output and spectrum. We summarize the more significant features below, referring to the appropriate figures and equations, the analytical details of whose derivation are available in Parts II and III.

1. **Signal Output (Power and Spectrum).** Spectrally speaking, the modulation is reproduced without distortion (2.32), since our model of the discriminator responds linearly to all frequencies about resonance. In practice, this is only approximate, because of unavoidable nonlinearities which result in harmonics of the modulation in the low-frequency output. The effect is small, however, if the maximum deviation remains mostly in the linear portion of the discriminator characteristic.

Figure 2 illustrates the output signal power $R_0(0)_{(s\times s)}$ [Eq. (2.32)] as a function of the input carrier-to-noise power ratio p. A significant difference between the limiting and non-limiting conditions of receiver operation is immediately evident. For no limiting ($\lambda = 1$) and sufficiently large values of p ($p^2 \gg 1$) the output signal power is directly proportional to p, while for extreme limiting ($\lambda = 2$) saturation is observed: the output signal strength is independent ($\backsim p^0$) of the carrier. The amount of noise is assumed to be constant in both instances.

On the other hand, the threshold behavior for weak signals ($p \lesssim 1$) shows that the signal output of the discriminator is proportional to p^2, whether or not there is limiting. Thus, if the carrier is sufficiently weak, the limiter-discriminator combination behaves like any (half-wave) second detector acting on an AM wave,[11] as far as the dependence on carrier strength is concerned. Similar remarks apply for the *d-c* output (Fig. 2),

[10]The integration is readily accomplished by representing exp $(-iR_0z)$ as the sum of two Bessel functions and evaluating the resulting Weber-Schafheitlin integrals with the help of Eq. (2), p. 401, Watson, *Theory of Bessel functions* (Macmillan, 1945). See also D. M. I., Appendix III.

[11]D. Middleton, *Rectification of a sinusoidally modulated carrier in the presence of noise*, Proc. I. R. E., **36**, 1467 (1948).

provided that we replace the mean-square modulation by the square of the mean. Accordingly, when the modulation possesses no d-c component, there will be no d-c output, [cf. (2.36), (2.37)]. From these equations and from physical considerations it is also obvious that if the modulation is slow enough, $\dot{\eta} \doteq 0$, there will be no observable signal.

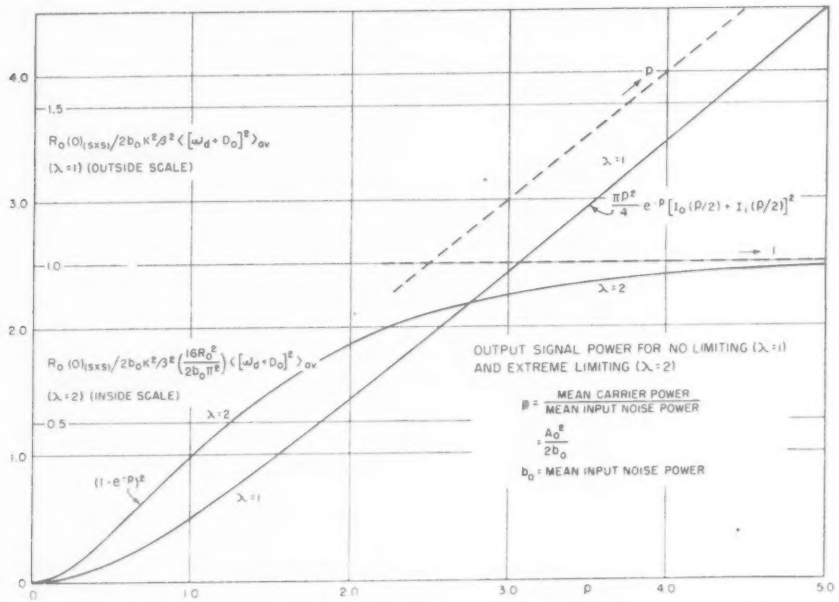

FIG. 2. Output signal power for no limiting ($\lambda = 1$) and extreme limiting ($\lambda = 2$).

2. Noise Output (Power). The low-frequency noise power from the discriminator is independent of the carrier when there is no limiting, provided the carrier is unmodulated [(Eq. (2.34)]. It is also true that the output does not depend on the modulating frequencies in either case, as can be verified from (2.34) and (2.35). However, unlike the analogous situation in AM reception[11] (mentioned above in connection with the signal) the magnitude of the noise power emanating from the discriminator *does* depend on the spectral distribution of the noise, i.e., on the IF-limiter-discriminator frequency response. The dependence is not very critical, covering a range of about 20-30 per cent between the extremes of rectangular and "optical," or single-tuned responses of the same energy (see Appendix V, and Table I of ref. 12).

Modulation suppression,[13] a phenomenon whereby the signal suppresses the noise when the signal is large enough, occurs in FM as well as in AM reception.[11] For no limiting ($\lambda = 1$) the low-frequency noise output is independent of carrier strength when $p^2 \gg 1$ and is still random noise, though mixed with components of the modulation [Eq. (2.40) et seq.]. A like behavior occurs in extreme limiting ($\lambda = 2$), except that

[12]D. Middleton, *Spurious signals caused by noise in triggered circuits*, J. Appl. Phys. **19**, 817 (1948).

[13]J. H. Van Vleck and D. Middleton, *A theoretical comparison of the visual, aural, and meter reception of pulsed signals in the presence of noise*, J. Appl. Phys. **17**, 940 (1946).

now the noise is absolutely, rather than relatively, suppressed, proportional to p^{-1}. The reverse situation arises when the carrier is weak relative to the noise. The noise is then dominant, and it is the signal which appears in the output as a perturbation of the noise. (See Sec. 1 above.)

3. Noise Output (Spectrum). It should be emphasized that discrimination, with or without limiting, is *not* a linear process, even though the noise output following it may be the linear sum of the effects of the transient pulses (and their derivatives) which make up the incoming noise wave. In general, from the spectral point of view the noise output after the nonlinear operations of limiting and discrimination will consist of three types of component, just as in conventional *AM* cases. These modulation products are produced by (a) beats between noise and noise (nxn), (b) beats between the signal and noise (sxn), and finally, (c) a signal contribution (sxs), due to products generated among the signal harmonics. For weak carriers the output spectrum is primarily determined by the distribution of the noise before discrimination (on the assumption of an ideal discriminator), while for strong carriers the spectrum of the discriminator output is not so heavily dependent on the spectral shape of the input, but is modified in a way that depends markedly on the degree of limiting. Limiting spreads the spectrum

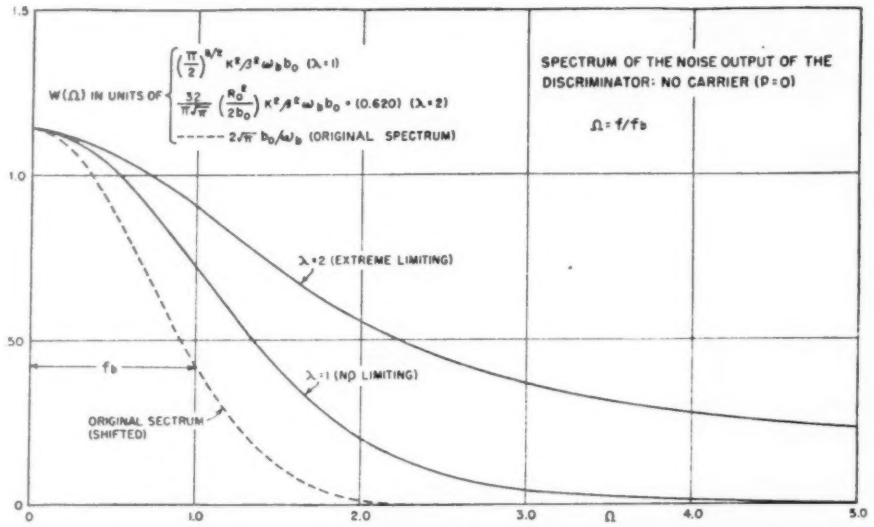

FIG. 3. Low-frequency noise spectrum following discrimination, no carrier.

by redistributing the energy into the higher harmonics of the distorted wave obtained in the clipping process.[14]

The significant feature of no limiting (strong carrier) is that the spectral intensity

[14]It has been implicitly assumed throughout that the limiter-discriminator filters are wide enough to pass the limited wave without appreciable distortion due to frequency selection. Thus, to consider spectra in the neighborhood of $\Omega = 5$, for example, is to assume these filters are at least five times wider than the *IF* band. The region of principal interest, however, is in most cases $\Omega < 2$, so that when there is limiting our model of a limiter-discriminator response several times as wide as the *IF* is not too far out of line with actual practice. For no limiting this response need not be wider than the *IF*.

near the low-frequency end of the output does not as a rule vanish. For extreme limiting, however, the intensity is *always* very small near $f = 0$, unlike the analogous situation in AM reception. Moreover, the spectrum near $f = 0$ is proportional to f^2 and is roughly triangular, becoming more nearly rectangular as the carrier diminishes in power relative to the noise, in accordance with Crosby's results.[2] Consequently, in *broad-band FM* the amount of noise passed in the video or audio stages is noticeably reduced when there is heavy limiting as compared with the instance of no limiting (Part III). Compare the spectra in Figs. 4, 5 and 9, for example. This shows the necessity for limiting if there is to be an improvement (over AM, and FM without limiting) in the signal-to-noise ratio when the carrier is strong.[2,3] Limiting, however, is not mandatory for weak signals, to which FM receivers respond like AM receivers in their dependence on carrier strength (see Sec. 1 above). For narrow-band FM the dependence on spectral distribution is not critical, since it is the entire low-frequency spectrum, rather than a fraction of it, which is a measure of the interfering noise. Spectral shape is then unimportant. Figures 6-11 illustrate various other special cases.

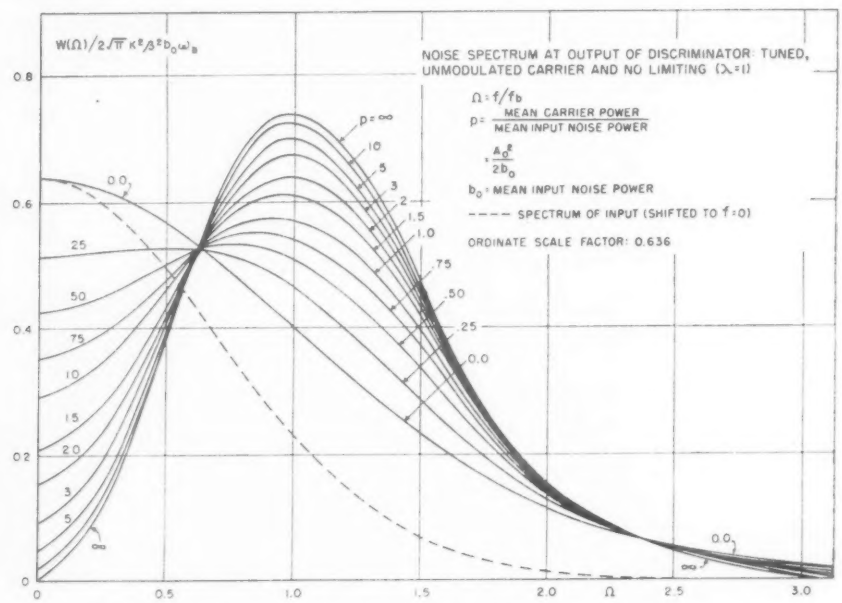

FIG. 4. Low-frequency noise spectrum following discrimination for a tuned, unmodulated carrier and no limiting.

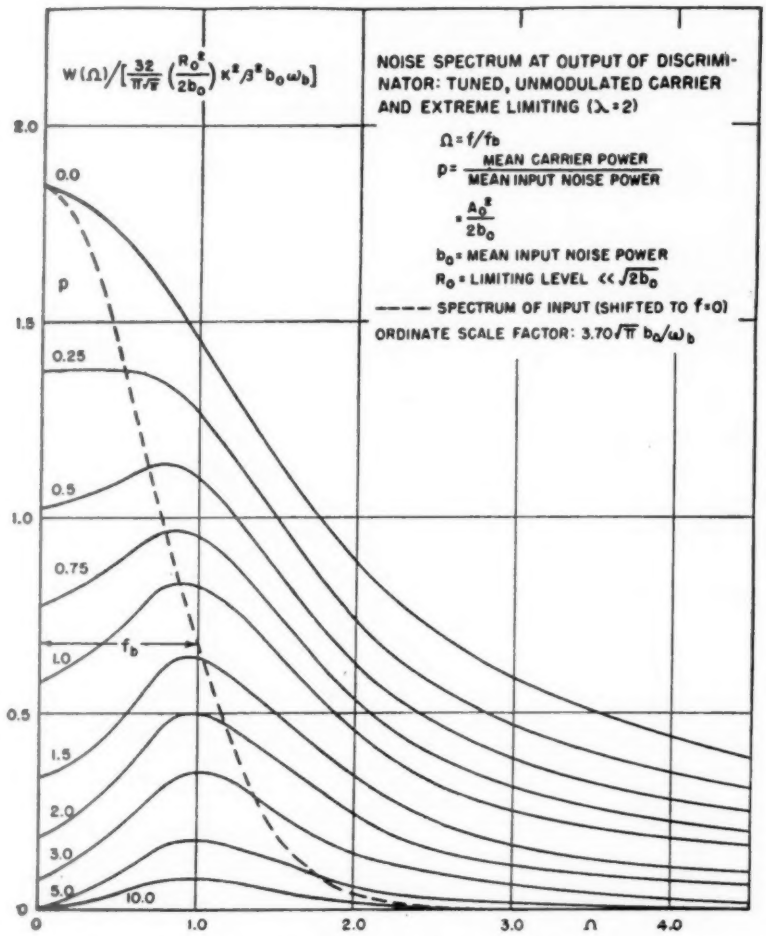

FIG. 5.* The same as Fig. 4, except now the limiting is extreme.

Correction added in proof: The ordinate scale factor in Fig. 5 is $2\pi^{1/2}b_0/1.85\omega_b = 1.08\,\pi^{1/2}b_0/\omega_b$.

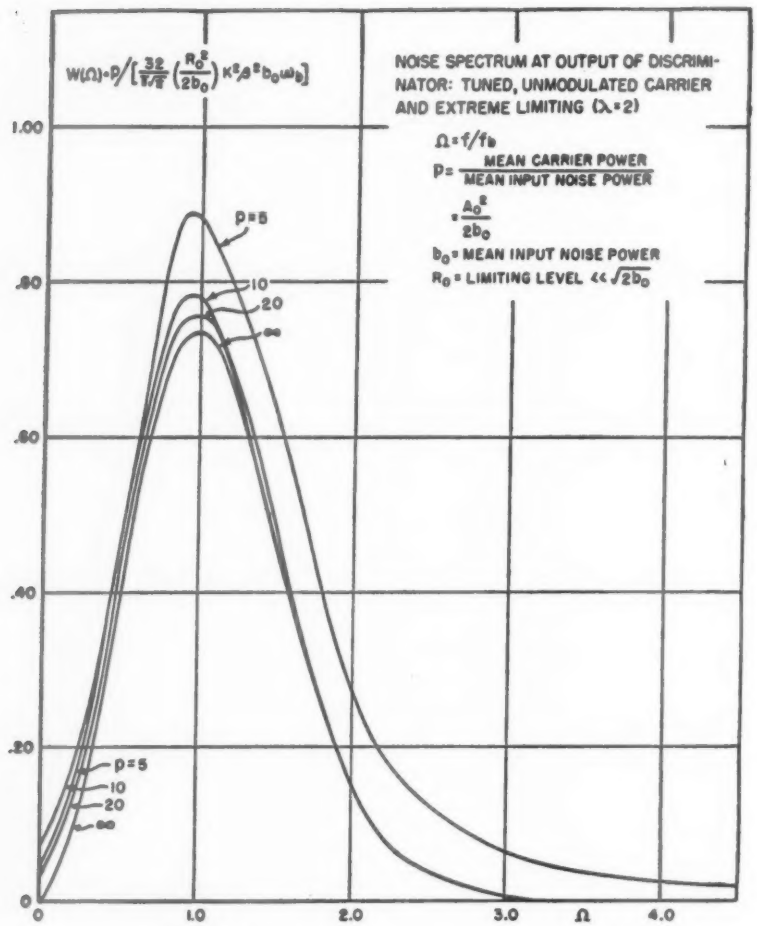

FIG. 6.* Figure 5, for large carrier amplitudes.

Correction added in proof: For $f < 0.3$, the curve for $p = 10$ in Fig. 6 should lie between the curves for $p = 5$ and $p = 20$.

Part II: Correlation Function of the Output (General Theory)

The correlation function of the low-frequency output of the discriminator is

$$R_0(t) = \overline{[E(t_0)E(t_0 + t)]_{av.}} = \overline{\kappa^2[B_1(R_1)B_1(R_2)\dot{\theta}_1\dot{\theta}_2]_{av.}} \,. \tag{2.1}$$

The bar indicates the statistical average over the random variables and $[\ ]_{av.}$ denotes the average over the phases of the modulation, if any.[15] The mean power spectrum is given by the well-known Wiener[16]-Khintchine[17] relation

$$W(f) = 4\int_0^\infty R(t)\cos\omega t\, dt, \qquad \omega = 2\pi f, \tag{2.2}$$

with the transform

$$R(t) = \int_0^\infty W(f)\cos\omega t\, df. \tag{2.2a}$$

The input voltage to our limiter-discriminator combination may be written

$$V_0(t) = V_s + V_N = A_0\cos(\omega_c t + \Psi) + \sum_{n=1}^{\infty}(a_n\cos\omega_n + b_n\sin\omega_n t), \tag{2.3}$$

where f_c is the carrier frequency (in the *IF* range) and

$$\Psi = \int^t D_0(t)\, dt, \qquad \text{with } D_0(t) \text{ the modulation.} \tag{2.3a}$$

The quantities a_n and b_n are random parameters with the properties

$$\overline{a_n} = \overline{b_m} = 0; \qquad \overline{a_n b_m} = 0; \qquad \overline{a_n a_m} = \overline{b_n b_m} = w(f_n)\Delta f\,\delta_m^n\,; \tag{2.4}$$

$w(f_n)\Delta f$ is the mean power dissipated in a unit resistance by the n-th component f_n,[18] and δ_m^n is the familiar Kronecker delta, such that $\delta_m^n = 0$, $n \neq m$, $\delta_n^n = 1$. Here A_0 is a constant representing the peak amplitude of the frequency-modulated carrier. Since the disturbance is assumed to be narrow-band, let

$$\omega_n = \omega_0 + \omega_n' \qquad \text{and} \qquad \omega_c = \omega_0 + \omega_d\,, \tag{2.5}$$

where ω_0 is some constant (angular) frequency, in this case the resonant or central frequency of the *IF*-limiter-discriminator elements; ω_n' is then a frequency relative to ω_0, and ω_d is simply the (fixed) deviation of the carrier from exact tuning ($\omega = \omega_0$). We have from (2.3) and (2.5)

$$V_N = \sum_{n=1}^{\infty}[a_n\cos(\omega_0 + \omega_n')t + b_n\sin(\omega_0 + \omega_n')t] = V_c\cos\omega_0 t - V_s\sin\omega_0 t, \tag{2.6}$$

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[15] The subscripts 1 and 2 on current or voltage amplitudes throughout refer to these amplitudes at the initial time $t_1 = t_0$ and a later time $t_2 = t_0 + t(\,t \geq 0)$, respectively.

[16] N. Wiener, Acta Math. **55**, 117 (1930).

[17] A. Khintchine, Math. Ann. **109**, 604 (1934).

[18] S. O. Rice, Bell Syst. Tech. J. **23**, 282 (1944) and M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. **17**, 323 (1945).

so that

$$V_c = \sum_{n=1}^{\infty} (a_n \cos \omega_n' t + b_n \sin \omega_n' t); \qquad V_s = \sum_{n=1}^{\infty} (a_n \sin \omega_n' t - b_n \cos \omega_n' t); \tag{2.7}$$

V_c and V_s are the portions of the slowly varying part of the wave that are in phase with $\cos \omega_0 t$ and $\sin \omega_0 t$, respectively. A similar treatment for the signal yields finally for the complete wave

$$\begin{aligned} V_0(t) &= [V_c + A_0 \cos (\omega_d t + \Psi)] \cos \omega_0 t - [V_s + A_0 \sin (\omega_d t + \Psi)] \sin \omega_0 t \\ &\equiv R \cos (\omega_0 t + \theta), \end{aligned} \tag{2.8}$$

so that

$$R = [(V_c + \alpha)^2 + (V_s + \beta)^2]^{1/2}, \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{V_s + \beta}{V_c + \alpha}\right) + n\pi \tag{2.9}$$

$$n = 0, 1, 2, \cdots,$$

in which

$$\alpha = A_0 \cos (\omega_d t + \Psi) \qquad \text{and} \qquad \beta = A_0 \sin (\omega_d t + \Psi). \tag{2.9a}$$

From (2.9) we obtain

$$\dot{\theta} = [(V_c + \alpha)(\dot{V}_s + \dot{\beta}) - (V_s + \beta)(\dot{V}_c + \dot{\alpha})]/R^2. \tag{2.10}$$

Consequently, the low-frequency output of our discriminator is [cf. (1.11)]

$$E_0(t) = \gamma_\lambda [(V_c + \alpha)(\dot{V}_s + \dot{\beta}) - (V_s + \beta)(\dot{V}_c + \dot{\alpha})]/R^\lambda, \tag{2.11}$$

where $\gamma_\lambda = \kappa\beta$ for $\lambda = 1$ (no limiting), and $\gamma_\lambda \doteq 4\kappa\beta R_0/\pi$ for $\lambda = 2$ (superlimiting). The correlation function (2.1) is therefore given by the eightfold integral

$$\begin{aligned} R_0(t)_N = \gamma_\lambda^2 \int_{-\infty}^{\infty} dV_{c1} \int_{-\infty}^{\infty} dV_{c2} \cdots \int_{-\infty}^{\infty} d\dot{V}_{s2} W_2(V_{c1}, \cdots, \dot{V}_{s2}; t) \\ \times [(V_{c1} + \alpha_1)(\dot{V}_{c1} + \dot{\beta}) - (V_{s1} + \beta_1)(\dot{V}_{c1} + \dot{\alpha}_1)] \\ \times [(V_{c2} + \alpha_2)(\dot{V}_{s2} + \dot{\beta}_2) - (V_{s2} + \beta_2)(\dot{V}_{c2} + \dot{\alpha}_2)]/R_1^\lambda R_2^\lambda . \end{aligned} \tag{2.12}$$

The subscript N indicates that only the average over the random variables $V_{c1}, \cdots, \dot{V}_{s2}$ has been performed, and not that over the phases of the modulation. The complete correlation function requires the additional average. The quantity W_2 is the joint probability density of the eight random variables $V_{c1}, \cdots, \dot{V}_{s2}$.

The evaluation of (2.12) will require the Fourier transform of W_2, namely, its characteristic function $F_2(z_1, z_2, \cdots, z_8; t)$, which is specifically[19]

[19]The subscripts 1, 2, $\cdots$, 8 here clearly do not have the time significance of those on $V_{c1}, V_{c2}, \cdots$; cf. footnote 15.

$$F_2(z_1, z_2, \cdots, z_8; t) = [\exp\{iz_1V_{c1} + iz_2V_{c2} + iz_3V_{s1} + iz_4V_{s2} + iz_5\dot{V}_{c1} + iz_6\dot{V}_{c2} + iz_7\dot{V}_{s1} + iz_8\dot{V}_{s2}\}]\text{stat. av.}$$

$$= \exp\left\{-\frac{b_0}{2}(z_1^2 + z_2^2 + z_3^2 + z_4^2) - \frac{b_2}{2}(z_5^2 + z_6^2 + z_7^2 + z_8^2) - \phi_0(t)(z_1z_2 + z_3z_4) - \phi_1(t)(z_1z_6 - z_2z_5 + z_3z_8 - z_4z_7) + \phi_2(t)(z_5z_6 + z_7z_8)\right\}, \quad (2.13)$$

where

$$b_n = \int_0^\infty w(f)(\omega - \omega_0)^n\, df, \qquad \text{and} \qquad \phi_n(t) = \frac{\partial^n}{\partial t^n}\int_0^\infty w(f)\cos(\omega - \omega_0)t\, df. \quad (2.14)$$

Since $w(f)$ is the mean input power spectrum, determined by the shape of the *IF* filter response, b_0 is the mean input noise power and $\phi_0(t)$ is the correlation function associated with it, by virtue of (2.2a). Observe that $\phi_0(0) = b_0$; then, for later use we find it convenient to write $\phi_n(t)$ in semi-normalized form, viz.

$$\phi_n(t)/b_0 \equiv r_n(t), \qquad \text{and} \qquad r_n(t)_{\max} \geq r_{n-1}(t)_{\max} \geq \cdots \geq r_0(t)_{\max} = 1, \quad (2.15)$$

as is easily shown. The determination of W_2 and F_2 is given in Appendix I. Our result (2.13) is somewhat specialized in that the input spectrum $w(f)$ is assumed to be symmetrical about f_0 .

Now, from the definition of the characteristic function and its Fourier transform relation with W_2 , the probability density may be expressed as

$$W_2(V_{c1}, V_{c2}, \cdots, \dot{V}_{s2}; t) = (2\pi)^{-8}\int_{-\infty}^{\infty} dz_1 \cdots \int_{-\infty}^{\infty} dz_8 \exp\{-i(z_1V_{c1} + z_2V_{c2} + \cdots + z_8\dot{V}_{s2})\}F_2(z_1, z_2, \cdots, z_8; t). \quad (2.16)$$

The correlation function (2.12) is then

$$R_0(t)_N = \gamma_\lambda^2\{I_1^{(\lambda)} - I_2^{(\lambda)} - I_3^{(\lambda)} + I_4^{(\lambda)}\}, \quad (2.17)$$

where

$$I_j^{(\lambda)} = \int_{-\infty}^{\infty} dV_{c1} \cdots \int_{-\infty}^{\infty} d\dot{V}_{s2}(2\pi)^{-8}\int_{-\infty}^{\infty} dz_1 \cdots \int_{-\infty}^{\infty} dz_8 F_2(z_1, \cdots, z_8; t)G_j(V_{c1}, \cdots, \dot{V}_{s2}) \times \exp\{-i(z_1V_{c1} + \cdots + z_8\dot{V}_{s2})\}R_1^{-\lambda}R_2^{-\lambda} \quad \cdot (j = 1, 2, 3, 4) \quad (2.18)$$

with

$$\begin{aligned} G_1 &= (V_{c1} + \alpha_1)(\dot{V}_{s1} + \dot{\beta}_1)(V_{c2} + \alpha_2)(\dot{V}_{s2} + \dot{\beta}_2), \\ G_2 &= (V_{c1} + \alpha_1)(\dot{V}_{s1} + \dot{\beta}_1)(V_{s2} + \beta_2)(\dot{V}_{c2} + \dot{\alpha}_2), \\ G_3 &= (V_{s1} + \beta_1)(\dot{V}_{c1} + \dot{\alpha}_1)(V_{c2} + \alpha_2)(\dot{V}_{s2} + \dot{\beta}_2), \\ G_4 &= (V_{s1} + \beta_1)(\dot{V}_{c1} + \dot{\alpha}_1)(V_{s2} + \beta_2)(\dot{V}_{c2} + \dot{\alpha}_2). \end{aligned} \tag{2.19}$$

At this point we follow a procedure suggested by some recent work of Rice.[4] We integrate first over the random variables V_{c1} and V_{s1}, V_{c2} and V_{s2}, taken in pairs, following which we perform the integration over the $\dot{V}$. We need the value of

$$K_1 \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \frac{x}{(x^2 + y^2)^{\mu}} \exp\{-ixz - iy\xi\},$$

and (2.20)

$$K_2 \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \frac{y}{(x^2 + y^2)^{\mu}} \exp\{-ixz - iy\xi\}.$$

Transformation to polar coordinates and integration give finally

$$K_1 = \frac{-2\pi i z}{\Gamma(\mu) 2^{2\mu-2}(z^2 + \xi^2)^{2-\mu}},$$

and (2.21)

$$K_2 = \frac{-2\pi i \xi}{\Gamma(\mu) 2^{2\mu-2}(z^2 + \xi^2)^{2-\mu}}, \qquad 2 > R(2\mu - 2) > -1.$$

For *superlimiting* λ is 2 and μ is unity, while in the other extreme of *no limiting* λ is 1 and μ is 1/2. For this latter case the integrals (2.20) do not converge, but the results (2.21) still apply, as Appendix II demonstrates. Now, letting $x = \alpha_1 + V_{c1}$, $y = \beta_1 + V_{s1}$, $\cdots$ and using (2.20), we obtain the twelve-fold integral

$$\begin{aligned} I_1^{(\lambda)} = &\int_{-\infty}^{\infty} d\dot{V}_{c1} \int_{-\infty}^{\infty} d\dot{V}_{c2} \cdots \int_{-\infty}^{\infty} d\dot{V}_{s2} (2\pi)^{-8} \int_{-\infty}^{\infty} dz_1 \\ &\cdots \int_{-\infty}^{\infty} dz_8 \exp\{-i(z_5\dot{V}_{c1} + \cdots + z_8\dot{V}_{s2})\} F_2(z_1, \cdots, z_8; t) \\ &\times \frac{(-2\pi i)^2 z_1 z_3 \exp\{i(z_1\alpha_1 + z_2\alpha_2 + z_3\beta_1 + z_4\beta_2)\}}{(z_1^2 + z_3^2)^{2-\lambda/2}(z_2^2 + z_4^2)^{2-\lambda/2}} (\dot{V}_{s1} + \dot{\beta}_1)(\dot{V}_{s2} + \dot{\beta}_2); \end{aligned} \tag{2.22}$$

$I_2^{(\lambda)}$, $I_3^{(\lambda)}$, and $I_4^{(\lambda)}$ are found from (2.22) if we replace $z_1 z_2(\dot{V}_{s1} + \dot{\beta}_1)(\dot{V}_{s2} + \dot{\beta}_2)$ by $z_1 z_4(\dot{V}_{s1} + \dot{\beta}_1)(\dot{V}_{c2} + \dot{\alpha}_2)$, $z_2 z_3(\dot{V}_{c1} + \dot{\alpha}_1)(\dot{V}_{s2} + \dot{\beta}_2)$, $z_3 z_4(\dot{V}_{c1} + \dot{\alpha}_1)(\dot{V}_{c2} + \dot{\alpha}_2)$ respectively. The integrations with respect to $\dot{V}_{c1}, \cdots, \dot{V}_{s2}$ are effected with the help of

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dY \int_{-\infty}^{\infty} \exp\{-izY\} g(z)\, dz = g(0),$$

and (2.23)

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} Y\, dY \int_{-\infty}^{\infty} \exp\{-izY\} g(z)\, dz = -i\left[\frac{\partial g}{\partial z}\right]_{z=0}.$$

The first relation follows from Fourier's integral theorem and the second also, on integration by parts; in both instances it is assumed that $g(\pm\infty)$ vanishes properly. Application of (2.23) in (2.22) for $I_1^{(\lambda)}$ yields the fourfold integral

$$I_1^{(\lambda)} = -(2\pi)^{-2}\int_{-\infty}^{\infty} dz_1 \cdots \int_{-\infty}^{\infty} dz_4$$

$$\times \frac{z_1 z_2\{\phi_1^2 z_3 z_4 - \phi_2 + i\dot{\beta}_1\phi_1 z_3 - i\dot{\beta}_2\phi_1 z_4 + \dot{\beta}_1\dot{\beta}_2\}}{(z_1^2+z_3^2)^{2-\lambda/2}(z_2^2+z_4^2)^{2-\lambda/2}} \tag{2.24}$$

$$\times \exp\left\{-\frac{b_0}{2}(z_1^2+z_2^2+z_3^2+z_4^2) - \phi_0(z_1z_2+z_3z_4) + i(\alpha_1 z_1 + \alpha_2 z_2 + \beta_1 z_3 + \beta_2 z_4)\right\}.$$

The other $I_j^{(\lambda)}$ are found from (2.24) if we replace its numerator, excluding the exponential, with

$$z_1 z_4\{\phi_1^2 z_1 z_4 + i\dot{\beta}_1\phi_1 z_1 - i\dot{\alpha}_2\phi_1 z_4 + \dot{\alpha}_2\dot{\beta}_1\},$$

$$z_2 z_3\{\phi_1^2 z_2 z_3 + i\dot{\alpha}_1\phi_1 z_3 - i\dot{\beta}_2\phi_1 z_2 + \dot{\alpha}_1\dot{\beta}_2\},$$

and

$$z_3 z_4\{(\phi_1^2 z_1 z_2 - \phi_2) + i\dot{\alpha}_1\phi_1 z_1 - i\dot{\alpha}_2\phi_1 z_2 + \dot{\alpha}_1\dot{\alpha}_2\},$$

respectively for $j = 2, 3, 4$. Returning to (2.17) with these expressions for $I_j^{(\lambda)}$, adding and collecting terms give us finally

$$R_0(t)_N = \frac{\dot{\gamma}_\lambda^2}{4\pi^2}\int_{-\infty}^{\infty} dz_1 \cdots \int_{-\infty}^{\infty} dz_4\{\phi_2(t)(z_1z_2+z_3z_4)$$

$$+ \phi_1(t)^2(z_1z_4 - z_2z_3)^2 - i\phi_1(t)(z_1z_4 - z_2z_3)(\dot{\alpha}_1 z_3 + \dot{\alpha}_2 z_4 - \dot{\beta}_1 z_1 - \dot{\beta}_2 z_2)$$

$$- (\dot{\beta}_1\dot{\beta}_2 z_1 z_2 + \dot{\alpha}_1\dot{\alpha}_2 z_3 z_4 - \dot{\alpha}_2\dot{\beta}_1 z_1 z_4 - \dot{\alpha}_1\dot{\beta}_2 z_2 z_3)\}(z_1^2+z_3^2)^{\lambda/2-2}(z_2^2+z_4^2)^{\lambda/2-2} \tag{2.25}$$

$$\times \exp\left\{-\frac{1}{2}b_0^2(z_1^2+z_2^2+z_3^2+z_4^2) - \phi_0(t)(z_1z_2+z_3z_4) + i(\alpha_1 z_1 + \alpha_2 z_2 + \beta_1 z_3 + \beta_2 z_4)\right\}.$$

At this point we make the obvious transformation to polar coordinates[20]

$$z_1 = \rho_1\cos\theta_1\,, \qquad z_2 = \rho_2\cos\theta_2\,,$$

$$z_3 = \rho_1\sin\theta_1\,, \qquad z_4 = \rho_2\sin\theta_2\,,$$

[20]When $\lambda = 2$ (superlimiting), we can follow an alternative method suggested by Rice[4] for the case ($\beta = 0, \alpha' = \beta' = 0$) in which the denominator $[(z_1^2+z_3^2)(z_2^2+z_4^2)]^{2-\lambda/2}$ is expressed as a pair of infinite integrals in (say) u and v, whose integrands have the form $\exp[-ub_0(z_1^2+z_3^2)/2]$ and $\exp[-vb_0(z_2^2+z_4^2)/2]$. The integration over $z_1\cdots z_4$ may be achieved with the help of a principal-axis transformation, leaving a sum of double integrals, each of which may be further reduced to a single one after suitable changes of variable. The final integration, however, cannot be effected in closed form, except in special cases. When there is no limiting ($\lambda = 1$), this technique fails, as it is not possible to eliminate the radicals $[(z_1^2+z_3^2)(z_2^2+z_4^2)]^{3/2}$.

for which the Jacobian is simply $\rho_1\rho_2$. From (2.3) and (2.3a) we may define a phase η by

$$\eta = \omega_d t + \Psi,$$

whence

$$\dot{\eta} = \omega_d + \dot{\Psi},$$

and $$\eta_1 = \omega_d t_0 + \Psi_1\,, \qquad \eta_2 = \omega_d(t_0 + t) + \Psi_2\,, \qquad t_2 \geq t_1 = t_0\,, \tag{2.26}$$

and since $(\alpha_{1,2}^2 + \beta_{1,2}^2)^{1/2} = A_0$ and $(\dot{\alpha}_{1,2}^2 + \dot{\beta}_{1,2}^2)^{1/2} = \dot{\eta}_{1,2}A_0$, we can write the correlation function in its general polar form

$$R_0(t)_N = \gamma_\lambda^2 \int_0^\infty \rho_1^{\lambda-2} \exp(-b_0\rho_1^2/2)\, d\rho_1 \int_0^\infty \rho_2^{\lambda-2} \exp(-b_0\rho_2^2/2)\, d\rho_2 \int_0^{2\pi} \frac{d\theta_1}{2\pi}$$

$$\times \int_0^{2\pi} \frac{d\theta_2}{2\pi} \exp\{-\phi_0\rho_1\rho_2 \cos(\theta_2 - \theta_1) + iA_0\rho_1 \cos(\theta_1 - \eta_1) + iA_0\rho_2 \cos(\theta_2 - \eta_2)\}$$

$$\times \{\phi_2 \cos(\theta_2 - \theta_1) + \phi_1^2\rho_1\rho_2 \sin^2(\theta_2 - \theta_1) + iA_0\phi_1 \sin(\theta_2 - \theta_1)[\dot{\eta}_1\rho_1 \cos(\theta_1 - \eta_1)$$

$$+ \dot{\eta}_2\rho_2 \cos(\theta_2 - \eta_2)] - A_0^2\dot{\eta}_1\dot{\eta}_2 \cos(\theta_1 - \eta_1) \cos(\theta_2 - \eta_2)\}. \tag{2.27}$$

The integration over θ_1 and θ_2 is accomplished in straight-forward fashion with the aid of the Bessel expansions of the various trigonometric exponentials [cf. (1.4)] and the orthonormality relations for the sine and cosine. The details are available in Appendix III. The correlation function becomes finally

$$\begin{aligned} R_0(t)_N = \gamma_\lambda^2 \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} r_0(t)^{2m+k}\Bigg\{\Bigg[& -\phi_2(t)\,\frac{\epsilon_k}{2}\Big(H^{(0)s}_{k,2m,k+1}\cos(k+1)(\eta_2-\eta_1) \\ & + H^{(0)s}_{k,2m,|k-1|}\cos(k-1)(\eta_2-\eta_1)\Big) \\ & + \phi_1(t)^2\,\frac{\epsilon_k}{2}\Big(H^{(1)s}_{k,2m+1,k}\cos k(\eta_2-\eta_1) - \frac{1}{2}H^{(1)s}_{k,2m+1,k+2}\cos(k+2)(\eta_2-\eta_1) \\ & - \frac{1}{2}H^{(1)s}_{k,2m+1,|k-2|}\cos(k-2)(\eta_2-\eta_1)\Big)\Bigg] \\ & + (2b_0p)^{1/2}\phi_1(t)\,\frac{(\dot{\eta}_1+\dot{\eta}_2)}{2}\Bigg[(1-\delta_0^k)H^{(0)}_{k,2m,k+1}(H^{(1)}_{k,2m+1,k+2} - H^{(1)}_{k,2m+1,k}) \\ & \times \sin(k+1)(\eta_2-\eta_1) \\ & - H^{(0)}_{k,2m,|k-1|}(H^{(1)}_{k,2m+1,k} - H^{(1)}_{k,2m+1,|k-2|})\sin|k-1|(\eta_2-\eta_1)\Bigg] \\ & + 2b_0p\dot{\eta}_1\dot{\eta}_2\cos k(\eta_2-\eta_1)\Bigg[\delta_0^k H^{(0)s}_{k,2m,k+1} + \frac{1}{2}(H^{(0)}_{k,2m,k+1} - H^{(0)}_{k,2m,|k-1|})^2\Bigg]\Bigg\}, \end{aligned} \tag{2.28}$$

where $p = A_0^2/2b_0$ is the ratio of the input mean-square carrier to mean-square noise

voltage. The amplitude functions $H^{(n)}_{k,2m+n,q}$, $q = |k-1|, k+1; q(n=0); q = k, k+2, |k-2|, (n=1)$ are obtained from

$$H^{(n)}_{k,2m+n,q}(p;\lambda) = \frac{b_0^{(2m+k)/2}}{2^{(2m+k)/2}[m!(m+k)!]^{1/2}} \times \int_0^\infty \rho^{2m+n+k+\lambda-2} \exp[-b_0\rho^2/2] J_q(A_0\rho)\, d\rho$$

$$= \frac{\Gamma[(k+2m+n+q+\lambda-1)/2] p^{q/2}}{q! b_0^{(n+\lambda-1)/2} 2^{(3-n-\lambda)/2}[m!(m+k)!]^{1/2}} \times {}_1F_1([k+2m+n+q+\lambda-1]/2; q+1; -p), \tag{2.29}$$

where k, m, n, q are integral and positive. The integration is performed with the help of[21]

$$\int_0^\infty J_\nu(ax) \exp(-b^2x^2) x^{\mu-1}\, dx = \frac{\Gamma[(\nu+\mu)/2]}{2b^\mu\, \Gamma(\nu+1)} \left(\frac{a}{2b}\right)^\nu {}_1F_1((\nu+\mu)/2; \nu+1; -a^2/4b^2), \tag{2.30}$$

$$R(\nu+\mu) > 0$$

in which ${}_1F_1$ is a confluent hypergeometric function. Recurrence relations for the various H's are available in Appendix IV.[22] The complete correlation function $R(t)$ follows after the average over the phases of the modulation has been taken, viz.

$$R_0(t) = \frac{1}{T_0} \int_0^{T_0} R_0(t)_N\, dt_{(mod)}\ . \tag{2.31}$$

The mean power spectrum may be obtained in the usual way [cf. (2.2)], by determining the Fourier transform of $R_0(t)$. Specific cases are considered in Part III.

As in the analogous situation for amplitude modulation,[11] the output will consist of three sets of contributions: (1) (nxn) noise generated by the beating of the input noise components with each other, (2) (sxn) noise produced by the cross-modulation of the signal and the noise, and finally, (3), the discrete signal components (sxs). In our expression (2.28) for the correlation function, only the quantities in the first bracket [ ] for which $q = 0$ [cf. (2.29)] represent (nxn) terms; the remaining ones of the first and all those in the second and third brackets [ ] represent (sxn) noise, except in the latter when $m = k = 0$; these are signal components:

$$R_0(t)_{(sxs)} = \gamma_\lambda^2 2b_0 p[\dot\eta_1\dot\eta_2]_{av} H^2_{001} = 2\gamma_\lambda^2 b_0 p H^2_{001}\{[\omega_d + D_0(t_0)][\omega_d + D_0(t_0+t)]\}_{av}. \tag{2.32}$$

[21] G. N. Watson, *Theory of bessel functions*, Cambridge University Press, 1945.

[22] In its present form the series (2.28), or its Fourier transform, does not converge very rapidly when $p < 1$, so that it is more convenient to expand the amplitude functions $H(p;\lambda)$ and collect coefficients of p^n $(n = 0, 1, 2, \cdots)$ before summing. Only a few such terms are then necessary when p is small, say 0.3 or less; interpolation between these values and those for $p \geq 1$ yields values of the correlation function and spectrum for the intermediate signal-to-noise ratios: $0.3 \leq p \leq 1$.

from (2.3a) and (2.28). Observe that the modulation is received without distortion since our idealized discriminator responds linearly to all frequencies.

The mean low-frequency power, W_0, may also be found from the correlation function on setting $t = 0$, according to a well-known theorem (D.M.I.). Now $\phi_1(0)$ vanishes, [cf. (2.14)], and $\eta_2 - \eta_1 = 0$, so that (2.28) reduces to

$$W_0 = R_0(0) = \overline{[E_0(t_0)^2]}_{av.} = \gamma_\lambda^2 \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left\{-\phi_2(0)\,\frac{\epsilon_k}{2}\right.$$
$$\times (H^2_{k,2m,k+1} + H^2_{k,2m,|k-1|}) + 2b_0 p\{[\omega_d + D_0(t_0)]^2\}_{av.}$$
$$\left.\times\left(\delta_0^k H^2_{k,2m,k+1} + \frac{1}{2}(H_{k,2m,k+1} - H_{k,2m,|k-1|})^2\right)\right\}, \tag{2.33}$$

of which $2b_0 p\gamma_\lambda^2 H^2_{001}\ \{[\omega_d + D_0(t_0)]^2\}_{av.}$ is the signal power; the rest is attributable to the noise. A direct calculation of $\overline{[E_0(t_0)^2]}_{av.} = K^2[B_1(R)^2\,\dot\theta^2]_{av.}$ using the first-order distribution density $W_1(R, \dot\theta)$ gives us[3,23] for no limiting,

$$(W_0)_{\lambda=1} = \kappa^2\beta^2\left\{-\phi_2(0) + 2b_0 p\left(1 - \frac{1 - e^{-p}}{2p}\right)\{[\omega_d + D_0(t_0)]^2\}_{av.}\right\}, \tag{2.34}$$

and for superlimiting

$$(W_0)_{\lambda=2} \doteq \frac{16}{\pi^2}\left(\frac{R_0^2}{2b_0}\right)\beta^2\kappa^2 e^{-p}\left\{(-\phi_2(0) + b_0 p\{[\omega_d + D_0(t_0)]^2\}_{av.})\right.$$
$$\times\left(\frac{\pi^2}{16} - 0.3444 - \mathrm{Ei}\left(-\frac{R_0^2}{2b_0}\right)\right) + \sum_{n=0}^{\infty}[(2n+3)p^{n+2}\{[\omega_d + D_0(t_0)]^2\}_{av.} \tag{2.35}$$
$$\left. - (n+2)\phi_2(0)p^{n+1}]/(n+1)(n+2)!\right\}, \qquad R_0^2 \ll 2b_0\,,$$

where $\mathrm{Ei}(-x) = -\int_x^\infty e^{-y}\,dy/y$. The exponential integral is tabulated in Jahnke and Emde's *Tables of Functions*,[24] (pp. 6-9). Note from (2.34) that for no carrier or for an unmodulated one that is on tune, i.e., $\omega_d = 0$, the mean noise power output ($\backsim -\phi_2(0)$) is independent of the carrier when there is no limiting. Then the spectral distribution of the noise, whatever its shape (for different values of p, $\omega_d = 0$) will always enclose the same area. Equation (2.35) shows us that such is not true when extreme limiting exists (except, of course when $p = 0$). It is also evident that the total output power is independent of the modulation frequencies, since $[D_0(t_0)^2]_{av.}$ does not depend on frequency.

The d-c output is easily found from our result (2.28) if we allow t to become infinite. There is then no correlation between the wave at time $t_1 = t_0$ and at a later time t. Since $\phi_n(t)$, $n \geq 0$, vanishes in the limit $t \to \infty$, only the term $m = k = 0$ independent of ϕ_1 and ϕ_2 will contribute to give us the square of the *d-c* component. Taking the square-root, for no limiting we have ($\lambda = 1$)

[23] D. Middleton, *On theoretical signal-to-noise ratios in FM receivers: a comparison with amplitude modulation*, to appear in J. Appl. Phys.

[24] E. Jahnke and F. Emde, *Tables of functions*, Dover Press, New York, 1943.

$$\left.\begin{aligned}\overline{[E_0(t_0)]}_{av.} &= \kappa\beta b_0^{1/2}[\omega_d + D_0(t_0)]_{av.}(\pi/2)^{1/2}p_1F_1(1/2;\,2;\,-p),\\ &= \kappa\beta b_0^{1/2}[\omega_d + D_0(t_0)]_{av.}(\pi/2)^{1/2}pe^{-p/2}[I_0(p/2) + I_1(p/2)].\end{aligned}\right\} \tag{2.36}$$

in terms of modified Bessel functions (see Appendix IV). The superlimiting case ($\lambda = 2$) yields

$$\left.\begin{aligned}\overline{[E_0(t_0)]}_{av.} &\doteq \frac{4}{\pi}\kappa\beta R_0[\omega_d + D_0(t_0)]_{av.}p\cdot{}_1F_1(1;\,2;\,-p),\\ &\doteq \frac{4}{\pi}\kappa\beta R_0[\omega_d + D_0(t_0)]_{av.}(1 - e^{-p}).\end{aligned}\right\} \tag{2.37}$$

Both (2.36) and (2.37) are in precise agreement with previous results.[3,23] If $[\omega_d + D_0(t_0)]_{av.}$ vanishes, there is no steady output component. (Figure 2 also illustrates the *square* of $\overline{[E_0(t_0)]}_{av.}$, provided we replace $\{[\omega_d + D_0(t_0)]^2\}_{av.}$ by $[\omega_d + D_0(t_0)]^2_{av.}$.)

We are now ready to examine the two important limiting cases which occur when the incoming carrier is strong, or weak, relative to the *rms* (input) noise level. Let us consider first the case when the carrier is strong, i.e., $p^2 \gg 1$. The asymptotic expansion of ${}_1F_1$ (see D.M.I.) is needed here, namely,

$${}_1F_1(\alpha;\,\beta;\,-x) \backsimeq \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\,x^{-\alpha}\left[1 + \frac{\alpha(\alpha-\beta+1)}{x} + \frac{\alpha(\alpha+1)(\alpha-\beta+1)(\alpha-\beta+2)}{2!x^2} + \cdots\right]. \tag{2.38}$$

1. Strong Carrier. For large ratios (p) we find accordingly that

$$H^{(n)}_{k,2m+n,q}(p;\,\lambda) \backsimeq \frac{\Gamma[(k+2m+n+q+\lambda-1)/2]p^{-(k+2m+n+\lambda-1)/2}}{b_0^{(n+\lambda-1)/2}2^{(3-n-\lambda)/2}[m!(m+k)!]^{1/2}} \times \frac{[1 + (k+2m+n+q+\lambda-1)(k+2m+n-q+\lambda-1)/4p + \cdots]}{\Gamma[(q+3-k-2m-n-\lambda)/2]}. \tag{2.39}$$

The correlation function (2.28) becomes

$$\begin{aligned}R_0(t) \backsimeq \frac{\gamma_s^2\Gamma(\lambda/2)^2p^{1-\lambda}}{b_0^{\lambda-1}2^{3-\lambda}\Gamma[(4-\lambda)/2]^2}\{&-\phi_2(t)[\cos(\eta_2-\eta_1)]_{av.}\\ &+ (\lambda-1)\phi_1(t)[(\dot\eta_1+\dot\eta_2)\sin(\eta_2-\eta_1)]_{av.}\\ &+ (\lambda-1)^2\phi_0(t)[\dot\eta_1\dot\eta_2\cos(\eta_2-\eta_1)]_{av.} + 2b_0p[\dot\eta_1\dot\eta_2]_{av.}\},\qquad p^2 \gg 1.\end{aligned} \tag{2.40}$$

The first three terms represent (sxn) noise, and the last term is the signal contribution (sxs); (nxn) noise is suppressed. Note that the contribution of the noise enters linearly in the correlation function: only ϕ_0 and its derivatives appear. This means that the output noise is still random in the "pure" sense in which it entered the limiter-discriminator elements. In other words, the effects of the transients which gave rise to the original noise in the *IF* still add linearly in the output. In spite of this, discrimination

remains a nonlinear process, as the mixing of noise and signal indicates (Eq. 2.40). That the noise can still remain random in the present case is due to the fact that discrimination is (for our ideal discriminator) frequency selective and not amplitude selective: the noise loses its "randomness" (normal properties) only when subject to amplitude limitation, i.e., distortion and/or clipping.

From (2.2) and with the help of (2.14) we may write the output noise spectrum when there is no limiting as

$$W_0(f)_{\lambda=1} \backsimeq -4\kappa^2\beta^2 \int_0^\infty \phi_2(t)[\cos(\eta_2 - \eta_1)]_{\text{av}} \cdot \cos\omega t\, dt, \qquad p^2 \gg 1,$$

$$\backsimeq 4\kappa^2\beta^2 \sum_{n=0}^{\infty} A_n \int_0^\infty \cos(n\omega_a + \omega_d)t \cos\omega t\, dt \int_0^\infty (\omega' - \omega_0)^2 \tag{2.41}$$

$$\times\, w(f') \cos(\omega' - \omega_0)t\, df', \qquad (f' = \dot{\omega}'/2\pi),$$

after expanding $[\cos(\eta_2 - \eta_1)]_{\text{av}}$ in a Fourier series; here $f_a(=\omega_a/2\pi)$ is the fundamental frequency of the modulation. Integration using the Dirac delta-function[6-8] gives us

$$W_0(f)_{\lambda=1} \backsimeq \kappa^2\beta^2 \sum_{n=0}^{\infty} A_n \int_0^\infty (\omega' - \omega_0)^2 w(f')[\delta(\omega' - \omega_0 - n\omega_a - \omega_d - \omega)$$

$$+ \delta(\omega' - \omega_0 - n\omega_a - \omega_d + \omega)]\, df' \tag{2.42}$$

$$\backsimeq \kappa^2\beta^2 \sum_{n=0}^{\infty} A_n[(n\omega_a + \omega_d + \omega)^2 w(\omega_0 + n\omega_a + \omega_d + \omega)$$

$$+ (n\omega_a + \omega_d - \omega)^2 w(\omega_0 + n\omega_a + \omega_d - \omega)].$$

Observe that the spectral density at and near $f = 0$ is finite and nonvanishing, no matter what distribution the input spectrum w has.

The case of superlimiting ($\lambda = 2$) when the carrier is strong is particularly interesting. If we let $x = t_0$ and $y = t_0 + t$, the three noise terms in (2.40) may be written

$$R_0(t)_{\text{noise}(\lambda=2)} \backsimeq \frac{16}{\pi^2}\kappa^2\beta^2\left(\frac{R_0^2}{2b_0}\right)p^{-1}\Big\{-\phi_2(t)\left[\cos\left(\omega_d t + \int_x^y D_0(t_0')\, dt'\right)\right]_{\text{av}} \cdot$$

$$+ \phi_1(t)\Big\{[2\omega_d + D_0(x) + D_0(y)]\sin\left(\omega_d t + \int_x^y D_0(t')\, dt'\right)\Big\}_{\text{av}} \cdot$$

$$+ \phi_0(t)\Big\{[\omega_d + D_0(x)][\omega_d + D_0(y)]\cos\left(\omega_d t + \int_x^y D_0(t_0')\, dt'\right)\Big\}_{\text{av}}\Big\} \tag{2.43}$$

$$\backsimeq \frac{16}{\pi^2}\kappa^2\beta^2\left(\frac{R_0^2}{2b_0}\right)p^{-1}\Big\{\left(-\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\right)\phi_0(t)$$

$$\times \int_0^{x_0} \cos\left[\omega_d t + \int_x^y D_0(t')\, dt'\right] dx/x_0\Big\},$$

where $x_0(=2\pi/\omega_a)$ is the period of the modulation and $\phi_n(t) \equiv (\partial^n/\partial t^n)\phi_0(t)$. Expansion of the integrand in a Fourier series, followed by the average over x, gives

$$\frac{1}{x_0}\int_0^{x_0} \cos\left[\omega_d t + \int_x^{y=x+t} D_0(t')\,dt'\right] dx = \sum_{n=0}^{\infty} A_n \cos(n\omega_a + \omega_d)(y-x), \qquad y - x = t, \tag{2.44}$$

since the result must be even in t. The coefficients A_n are the same as those for the development in (2.41). Next, we substitute (2.44) into (2.43), which shows that

$$R_0(t)_{\text{noise}(\lambda=2)} \backsimeq \frac{16}{\pi^2}\kappa^2\beta^2\left(\frac{R_0^2}{2b_0}\right)p^{-1}\sum_{n=0}^{\infty} A_n[-\phi_2(t)\cos(n\omega_a+\omega_d)t + 2(n\omega_a+\omega_d)\phi_1(t)\sin(n\omega_a+\omega_d)t + (n\omega_a+\omega_d)^2\phi_0(t)\cos(n\omega_a+\omega_d)t]. \tag{2.45}$$

Use of the integral form of $\phi_n(t)$ gives us, finally, a relation similar to (2.41) for the noise spectrum:

$$W_0(f)_{\lambda=2} \backsimeq \frac{64}{\pi^2}\kappa^2\beta^2\left(\frac{R_0^2}{2b_0}\right)p^{-1}\sum_{n=0}^{\infty} A_n\Big\{\int_0^{\infty}\cos(n\omega_a+\omega_d)t\cos\omega t\,dt \times \int_0^{\infty}[(\omega'-\omega_0)^2 + (n\omega_a+\omega_d)^2]w(f')\cos(\omega'-\omega_0)t\,df' \tag{2.46}$$

$$-2\int_0^{\infty}(n\omega_a+\omega_d)\sin(n\omega_a+\omega_d)t\cos\omega t\,dt\int_0^{\infty}(\omega'-\omega_0)w(f')\sin(\omega'-\omega_0)t\,df'\Big\},$$

$$W_0(f)_{\lambda=2} \backsimeq \frac{16}{\pi^2}\kappa^2\beta^2\left(\frac{R_0^2}{2b_0}\right)p^{-1}\sum_{n=0}^{\infty} A_n\Big\{\int_0^{\infty}[(\omega'-\omega_0)^2 + (n\omega_a+\omega_d)^2 - 2(n\omega_a+\omega_d) \times (\omega'-\omega_0)]w(f')[\delta(\omega'-\omega_0-n\omega_a-\omega_d+\omega) + \delta(\omega'-\omega_0-n\omega_a-\omega_d-\omega)]\,df'\Big\}$$

which is

$$W_0(f)_{\lambda=2} \backsimeq \frac{16}{\pi^2}\kappa^2\beta^2\left(\frac{R_0^2}{2b_0}\right)p^{-1}\sum_{n=0}^{\infty} A_n\omega^2[w(\omega_0+n\omega_a+\omega_d+\omega) + w(\omega_0+n\omega_a+\omega_d-\omega)], \qquad p^2 \gg 1. \tag{2.47}$$

The significant fact about (2.47) is that the spectral intensity for superlimiting always vanishes at $f = 0$ and is small in the vicinity of $f = 0$. Compared with our result for the unlimited case [cf. (2.42)], this shows why *for a sufficiently strong carrier, limiting gives a noticeable improvement over no limiting, provided we observe the signal component(s) in a band, near zero frequency, which is narrow relative to the width* ($\backsim f_b$) *of the (combined) IF-limiter-discriminator filter response*, i.e., $f \ll f_b$. This result is identical with Blachman's,[3] which was derived by a different method, and it agrees with the earlier work of Carson and Fry.[1]

From (2.40) we observe that for $p^2 \gg 1$ the noise is always suppressed, its contribution being proportional to $p^{1-\lambda}$ and that of the signal to $p^{2-\lambda}$. Accordingly, when

there is no limiting, the noise power is constant and the output signal strength increases as p^1, whereas for extreme limiting, the noise actually vanishes as $p \to \infty$, and the signal reaches a limiting value independent of p. These comments are illustrated in Fig. 2. We notice also from (2.40) that when $\lambda = 1$, the noise power output is independent of modulation; this is not true for extreme limiting ($\lambda = 2$).

2. Weak Signals. When the signal is weak or, at most, of the same order of magnitude as the noise, the correlation function (2.28) is most conveniently described[22] by a power series in p; if we use (2.29) and omit terms in $p^n (n \geq 3)$, we obtain finally after considerable reduction the following result:

$$R_0(t) = \gamma_\lambda^2 \Gamma(\lambda/2)^2 2^{\lambda-3} b_0^{2-\lambda} \{G_0(t) + pG_1(t) + p^2[G_2(t) + 2\{\dot{\eta}_1\dot{\eta}_2\}_{av.}] + \cdots\} \qquad p^3 \ll 1, \tag{2.48}$$

where

$$G_0(t) = (r_1^2 - r_0 r_2)\ {}_2F_1(\lambda/2, \lambda/2; 2; r_0^2), \tag{2.49a}$$

$$\begin{aligned} G_1(t) = (r_1^2 - r_0 r_2)\Big\{ & -\lambda\ {}_2F_1(\lambda/2, \lambda/2 + 1; 2; r_0^2) \\ & + r_0 \frac{\lambda^2}{4}\ {}_2F_1(\lambda/2 + 1, \lambda/2 + 1; 3; r_0^2)[\cos(\eta_2 - \eta_1)]_{av.}\Big\} + \Big\{ r_0[\dot{\eta}_1\dot{\eta}_2 \cos(\eta_2 - \eta_1)]_{av.} \\ & - r_1[(\dot{\eta}_1 + \dot{\eta}_2)\sin(\eta_2 - \eta_1)]_{av.} - r_2[\cos(\eta_2 - \eta_1)]_{av.}\Big\}\ {}_2F_1(\lambda/2, \lambda/2; 2; r_0^2), \end{aligned} \tag{2.49b}$$

and

$$\begin{aligned} G_2(t) = (r_1^2 - r_0 r_2)\Big\{ & \frac{\lambda^2}{4}\ {}_2F_1(\lambda/2 + 1, \lambda/2 + 1; 2; r_0^2) + \lambda/4(\lambda/2 + 1) \cdot {}_2F_1(\lambda/2, \lambda/2 + 2; 2; r_0^2) \\ & - \frac{r_0\lambda^2}{4}(\lambda/2 + 1)\ {}_2F_1(\lambda/2 + 1, \lambda/2 + 2; 3; r_0^2)[\cos(\eta_2 - \eta_1)]_{av.} \\ & + \frac{\lambda^2}{16}[\cos 2(\eta_2 - \eta_1)]_{av.}[2\ {}_2F_1(\lambda/2 + 1, \lambda/2 + 1; 2; r_0^2) - {}_2F_1(\lambda/2 + 1, \\ & \lambda/2 + 1; 3; r_0^2)]\Big\} + {}_2F_1(\lambda/2, \lambda/2 + 1; 2; r_0^2)\Big\{ -\frac{3\lambda}{2} r_0[\dot{\eta}_1\dot{\eta}_2 \cos(\eta_2 - \eta_1)]_{av.} \\ & + \lambda r_1[(\dot{\eta}_1 + \dot{\eta}_2)\sin(\eta_2 - \eta_1)]_{av.} + r_2\,\lambda/2[\cos(\eta_2 - \eta_1)]_{av.}\Big\} \\ & + {}_2F_1(\lambda/2 + 1, \lambda/2 + 1; 3; r_0^2)\Big\{ \frac{-r_0 r_1 \lambda^2}{8}[(\dot{\eta}_1 + \dot{\eta}_2)\sin 2(\eta_2 - \eta_1)]_{av.} \\ & + \frac{r_0^2\lambda^2}{8}[\dot{\eta}_1\dot{\eta}_2 \cos 2(\eta_2 - \eta_1)]_{av.} - \frac{r_1^2\lambda^2}{8}[\cos 2(\eta_2 - \eta_1)]_{av.}\Big\} \\ & + 2[\dot{\eta}_1\dot{\eta}_2]_{av.}[{}_2F_1(\lambda/2, \lambda/2; 1; r_0^2) - 1]. \end{aligned} \tag{2.49c}$$

The reduction is accomplished with the aid of the recurrence relation

$$ {}_2F_1(a, b; c; x) - {}_2F_1(a, b; c+1; x) = \frac{abx}{c(c+1)} {}_2F_1(a+1, b+1; c+2; x). \tag{2.50} $$

We see at once from (2.48) and (2.49) that the output noise is no longer "pure," i.e., the transients producing the noise no longer add linearly in the output, unlike the case of a strong carrier and weak noise.

The first term in (2.48) represents the correlation function for the noise output that would exist in the absence of a carrier, and $2p^2[\eta_1\dot{\eta}_2]_{av}$. is the expression for the output signal or modulation. *In threshold perception* ($p \lesssim 1$) *the observed signal is suppressed by the noise and the output signal-to-noise (power) ratio is therefore directly proportional to* p^2 *(for the leading term), just as in the amplitude modulated cases.*[11,13*] This is true whether or not there is limiting.[25] Figure 2 shows the output signal power as a function of the input carrier-to-noise power.

Part III: Special Cases

1. Noise alone. The first important example occurs when the demodulated wave is narrow-band noise. Here $\eta_1 = \eta_2 = \dot{\eta}_1 = \dot{\eta}_2 = 0$ and $p = 0$ also, so that the correlation function (2.48) is simply.

$$ \begin{aligned} R_0(t) &= \gamma_\lambda^2(r_1^2 - r_0r_2)b_0^{2-\lambda}2^{\lambda-3}\sum_{m=0}^{\infty}\frac{r_0^{2m}\Gamma(m+\lambda/2)^2}{m!(m+1)!}, \\ &= \gamma_\lambda^2(r_1^2 - r_0r_2)b_0^{2-\lambda}2^{\lambda-3}\Gamma(\lambda/2)^2\ {}_2F_1(\lambda/2, \lambda/2; 2; r_0^2). \end{aligned} \tag{3.1} $$

In the instance of no limiting we obtain

$$ \begin{aligned} R_0(t)_{\lambda=1} &= \frac{\pi}{4}\kappa^2\beta^2 b_0(r_1^2 - r_0r_2)\ {}_2F_1(1/2, 1/2; 2; r_0^2), \\ &= \kappa^2\beta^2\frac{b_0}{2}(r_1^2 - r_0r_2)K(r_0^2), \end{aligned} \tag{3.2} $$

where $K(r_0^2)$ is an elliptic integral of the first kind, with modulus r_0 . With the aid of

$$ {}_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, $$

$$ R(\gamma) \neq 0, -1, -2, \qquad R(\gamma-\alpha-\beta) \neq 0, -1, -2, \cdots, $$

we find the mean low-frequency power to be

$$ (W_0)_{\lambda=1} = R_0(0)_{\lambda=1} = -\kappa^2\beta^2\phi_2(0) = \kappa^2\beta^2\int_0^\infty(\omega-\omega_0)^2 w(f)\,df, \tag{3.3} $$

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*See, in particular, Sec. VII, Part I and Sec. II, Part II, ref. 13.

[25]See references 23 and 3 for a discussion of signal-to-noise ratios.

a special case of (2.33) or (2.34) when $p = 0$. The spectrum follows from (2.2) and (3.1); it is

$$W_0(f)_{\lambda=1} = \pi b_0 \kappa^2 \beta^2 \sum_{m=0}^{\infty} \frac{(1/2)_m^2}{m!(2)_m} \int_0^{\infty} [r_1(t)^2 - r_0(t) r_2(t)] r_0^{2m}(t) \cos \omega t \, dt. \tag{3.4}$$

At this point, and for all subsequent calculations, we assume a Gaussian spectral distribution for the incoming noise, representing the composite *IF*-limiter-discriminator frequency response, viz.,

$$w(f) = W_0 \exp\{-(\omega - \omega_0)^2/\omega_b^2\}, \qquad \text{and} \qquad b_0 = W_0 \omega_b / 2\pi^{1/2}, \tag{3.5}$$

where W_0 is the maximum spectral intensity of the noise entering the limiter. From (2.14) and (2.15) we find that

$$r_0(t) = \exp\{-\omega_b^2 t^2/4\}, \qquad r_1(t) = -\omega_b^2 t r_0(t)/2, \qquad r_2(t) = \omega_b^2(\omega_b^2 t^2 - 2) r_0(t)/4, \tag{3.6}$$

so that $(r_1^2 - r_0 r_2) = \omega_b^2 r_0(t)^2/2$. Similar relations for the "optical" or single-tuned characteristic and for rectangular spectra are discussed briefly in Appendix V. Substitution of (3.6) into (3.4) gives

$$W_0(f)_{\lambda=1} = \left(\frac{\pi}{2}\right)^{3/2} \kappa^2 \beta^2 \omega_b b_0 \sum_{m=0}^{\infty} \frac{(1/2)_m^2}{(m!)^2 (m+1)^{3/2}} \exp\{-\Omega^2/(2m+2)\}, \tag{3.7}$$

$$\text{where } \Omega \equiv \omega/\omega_b = f/f_b \, .$$

The distribution is shown in Fig. 3.

The correlation function for extreme limiting assumes the form

$$\begin{aligned} R_0(t)_{\lambda=2} &\doteq \frac{16}{\pi^2} \beta^2 \kappa^2 \left(\frac{R_0^2}{2b_0}\right) b_0 (r_1^2 - r_0 r_2) \, {}_2F_1(1, 1; 2; r_0^2) \\ &= \frac{-16}{\pi^2} \beta^2 \kappa^2 \left(\frac{R_0^2}{2b_0}\right) b_0 \left(\frac{r_1^2 - r_0 r_2}{r_0^2}\right) \log_e (1 - r_0^2). \end{aligned} \tag{3.8}$$

Although $R_0(0)_{(\lambda=2)}$ is logarithmically divergent, the mean (low-frequency) output power is actually not infinite, but zero, since (3.8) applies rigorously when $t \to 0$ only if $R_0^2/2b_0 \to 0$ also; in other words, the limiting is so extreme that essentially no low-frequency components are produced. All the energy goes into higher harmonics.[6] For computation of spectra, however, it is sufficient that $R_0^2/2b_0 \ll 1$.

We obtain the spectrum ($\lambda = 2$) at once from (3.1) or (3.8) and (2.2) for the Gaussian distribution (3.5). We have

$$W_0(f)_{\lambda=2} \doteq \frac{32}{\pi(2\pi)^{1/2}} \left(\frac{R_0^2}{2b_0}\right) \kappa^2 \beta^2 \omega_b b_0 \sum_{m=0}^{\infty} \frac{\exp\{-\Omega^2/(2m+2)\}}{(m+1)^{3/2}}. \tag{3.9}$$

Figure 3 also illustrates $W_0(f)_{\lambda=2}$ as a function of Ω. The spectral intensity at $\Omega = 0$ is

$$W_0(0)_{\lambda=2} \doteq \frac{32}{\pi(2\pi)^{1/2}} \left(\frac{R_0^2}{2b_0}\right) \beta^2 \kappa^2 \omega_b b_0 \zeta(3/2), \tag{3.10}$$

where ζ is Riemann's zeta-function[26] and $\zeta(3/2)$ has the value 2.612. Some details of the evaluation of (3.7) and (3.9) are available in Appendix V. From (A5-2) it also

[26] Jahnke and Emde, *loc. cit.*, pp. 269-273.

follows that when Ω becomes infinite, the spectral intensity (3.9) falls to zero as $W_0(f)_{\lambda=2} \asymp 16\beta^2\kappa^2\omega_b R_0^2/\pi\,\Omega(\Omega \to \infty)$.

2. Tuned, unmodulated carrier. The unmodulated carrier in noise is the next more complicated case. Now $\eta_1 = \eta_2 = 0$ and $\dot{\eta}_1 = \dot{\eta}_2 = 0$ also, provided the carrier is tuned to the center of the band, i.e., $\omega_d = 0$. The correlation function (2.28) then becomes

$$R_0(t) = \gamma_\lambda^2 \sum_{k=0}^{\infty} \frac{\epsilon_k}{2} \sum_{m=0}^{\infty} r_0(t)^{2m+k}\Big\{-\phi_2(t)(H^2_{k,2m,k+1} + H^2_{k,2m,|k-1|}) + \phi_1(t)^2\Big(H^2_{k,2m+1,k} - \frac{1}{2}H^2_{k,2m+1,k+2} - \frac{1}{2}H^2_{k,2m+1,|k-2|}\Big)\Big\} \tag{3.11}$$

with $p = A_0^2/2b_0$ as before. The power output associated with the low-frequency terms is obtained at once from (2.34) and (2.35), viz:

$$(W_0)_{\lambda=1} = -\kappa^2\beta^2\phi_2(0), \tag{3.12}$$

$$(W_0)_{\lambda=2} \doteq -\frac{16}{\pi^2}\Big(\frac{R_0^2}{2b_0}\Big)\beta^2\kappa^2\phi_2(0)e^{-p}\Big\{\frac{\pi^2}{16} - Ei\Big(-\frac{R_0^2}{2b_0}\Big) - 0.3444 + \sum_{n=0}^{\infty}\frac{p^{n+1}}{(n+1)!(n+1)}\Big\}.$$

Observe that the mean-square low-frequency output of the ideal discriminator is independent of carrier strength when there is no limiting. This is what one would expect, since the carrier is tuned to the central frequency of the symmetrical band, and its *d-c* therefore vanishes.

The mean power spectrum follows as before from (2.2) and (3.11). For the Gaussian input noise distribution it is

$$W_0(f) = \gamma_\lambda^2 2\pi^{1/2}\omega_b b_0\Big\{\sum_{k=0}^{\infty}\frac{\epsilon_k}{2}\sum_{m=0}^{\infty}\Big[\Big(2m+k+\frac{2\Omega^2}{2m+k+2}\Big)(H^2_{k,2m,k+1} + H^2_{k,2m,|k-1|})\frac{\exp\{-\Omega^2/(2m+k+1)\}}{(2m+k+1)^{3/2}} + b_0\Big(1 - \frac{2\Omega^2}{2m+k+2}\Big) \times \Big(H^2_{k,2m+1,k} - \frac{1}{2}H^2_{k,2m+1,k+2} - \frac{1}{2}H^2_{k,2m+1,|k-2|}\Big)\frac{\exp\{-\Omega^2/(2m+k+2)\}}{(2m+k+2)^{3/2}}\Big]\Big\}. \tag{3.13}$$

The spectra for no limiting are shown in Fig. 4 for various values of p, and for superlimiting in Figs. 5 and 6. The significance of the results is discussed briefly in Part I.

When the carrier is large, the spectrum is found at once from (2.42) or (2.47). These results give the leading term, while a further use of the asymptotic development (2.39) gives the succeeding one. Here

$$A_n = \delta_n^0$$

and $$w(f) = 2b_0\omega_b^{-1}\pi^{1/2}\exp\{-(\omega-\omega_0)^2/\omega_b^2\}, \qquad \text{so that}$$

$$W_0(f)_{\lambda=1} \asymp 4\pi^{1/2}\kappa^2\beta^2 b_0\omega_b\Big\{\Omega^2\exp(-\Omega^2) - \frac{1}{2p}\Big[\Omega^2\exp(-\Omega^2) - \frac{1}{4\cdot 2^{1/2}}(1+\Omega^2)\exp\{-\Omega^2/2\}\Big] + \cdots\Big\}, \; p^2 \gg 1. \tag{3.14}$$

A similar procedure for superlimiting [cf. (2.47)] yields

$$W_0(f)_{\lambda=2} \simeq \frac{64}{\pi^{3/2}} \kappa^2\beta^2\left(\frac{R_0^2}{2b_0}\right) b_0\omega_b \frac{1}{p}\left\{\Omega^2 \exp(-\Omega^2) + \frac{\exp\{-\Omega^2/2\}}{2^{3/2}p} + \cdots\right\}. \tag{3.15}$$

The mean output power corresponding to (3.12) when $p^2 \gg 1$ is simply $-\kappa^2\beta^2\phi_z(0)$ when $\lambda = 1$ and is

$$(W_0)_{\lambda=2} = \int_0^\infty W_0(f)_{\lambda=2}\, df \simeq \frac{8}{\pi^2}(\kappa\beta\omega_b)^2\left(\frac{R_0^2}{2b_0}\right)\frac{b_0}{p}\left\{1 + \frac{2^{-7/4}}{p} + \cdots\right\}. \tag{3.16}$$

We notice that when there is extreme limiting ($R_0^2 \ll 2b_0$) and the carrier is strong ($p^2 \gg 1$), suppression of the noise occurs, analogous to the similar situation in the reception of amplitude-modulated waves.[11,13] The spectra for the strong carrier case are illustrated in Fig. 6.

The small-signal problem, on the other hand, gives us a different output spectrum. From (2.2), (2.48), and (2.49) the spectrum of the noise may be written

$$W_0(f) \doteq \gamma_\lambda^2\Gamma(\lambda/2)^2 2^{\lambda-1} b_0^{2-\lambda}\int_0^\infty [G_0(t) + pG_1(t) + p^2G_2(t) + \cdots]\cos\omega t\, dt, \tag{3.17}$$

which in the instance of the Gaussian distribution (3.5) becomes finally

$$\begin{aligned}
W_0(f) \doteq\ & W_0(f)_{\text{noise}}p^0 + \gamma_\lambda^2\Gamma(\lambda/2)^2\omega_b\pi^{1/2}2^{\lambda-2}b_0^{2-\lambda} \\
& \times\left\{p\left[\sum_{m=0}^{\infty}\frac{-\lambda(\lambda/2)_m(\lambda/2+1)_m\exp\{-\Omega^2/(2m+2)\}}{m!(m+1)!(2m+2)^{1/2}}\right.\right. \\
& + \frac{2(\lambda/2)^2(\lambda/2+1)_m^2\exp\{-\Omega^2/(2m+3)\}}{m!(m+2)!(2m+3)^{1/2}} \\
& - \frac{(\lambda/2)_m^2}{m!(m+1)!(2m+1)^{1/2}}\left[\left(1 - \frac{2\Omega^2}{2m+1}\right)\frac{1}{(2m+1)}\right. \\
& \left.\left. - 1\right]\exp\{-\Omega^2/(2m+1)\}\right] \\
& + p^2\left[\sum_{m=0}^{\infty}{}'\left(\left[\frac{\lambda^2}{4}\left(\frac{\lambda}{2}+1\right)_m^2 + \frac{\lambda}{4}\left(\frac{\lambda}{2}+1\right)\left(\frac{\lambda}{2}\right)_m\left(\frac{\lambda}{2}+2\right)_m\right.\right.\right. \\
& \left.\left.\left. + \frac{1}{2}\left(\frac{\lambda}{2}\right)^2\left(\frac{\lambda}{2}+1\right)_m^2\left(\frac{m+1}{m+2}\right)\right]\right. \\
& \times\frac{\exp\{-\Omega^2/(2m+2)\}}{m!(m+1)!(2m+2)^{1/2}} \\
& - 2\frac{(\lambda/2)^2(\lambda/2+1)(\lambda/2+1)_m(\lambda/2+2)_m\exp\{-\Omega^2/(2m+3)\}}{m!(m+2)!(2m+3)^{1/2}} \\
& - \frac{(\lambda/2)(\lambda/2)_m(\lambda/2+1)_m}{m!(m+1)!(2m+1)^{1/2}}\left[\left(1 - \frac{2\Omega^2}{2m+1}\right)\frac{1}{(2m+1)} - 1\right]\exp\{-\Omega^2/(2m+1)\} \\
& \left.\left. - \frac{(\lambda/2)^2(\lambda/2+1)_m^2}{m!(m+2)!(2m+2)^{3/2}}\left(1 - \frac{\Omega^2}{m+1}\right)\exp\{-\Omega^2/(2m+2)\}\right] + \cdots\right\}.
\end{aligned} \tag{3.18}$$

The curves of Figs. 4 and 5 for $p < 0.3$ have been calculated with the help of (3.18).[22]

3. Off-tune, unmodulated carrier. This is a generalization of the preceding case to include the effects of a constant deviation of the carrier from the *IF*-limiter-discriminator resonant frequency f_0. It is now evident that $\eta_1 = \omega_d t_0$, $\eta_2 = \omega_d(t_0 + t)$, and so $\dot{\eta}_1 = \dot{\eta}_2 = \omega_d$. The correlation function (2.28) reduces to

$$
\begin{aligned}
R_0(t) = \gamma_\lambda^2 \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} r_0^{2m+k} \Bigg\{ & -\phi_2 \frac{\epsilon_k}{2} \Big(H^2_{k,2m,k+1} \cos (k+1)\omega_d t \\
& + H^2_{k,2m,|k-1|} \cos (k-1)\omega_d t \Big) \\
& + \phi_1^2 \frac{\epsilon_k}{2} \Big(H^2_{k,2m+1,k} \cos k\omega_d t - \frac{1}{2} H^2_{k,2m+1,k+2} \cos (k+2)\omega_d t \\
& - \frac{1}{2} H^2_{k,2m+1,|k-2|} \cos (k-2)\omega_d t \Big) \\
& + (2b_0 p)^{1/2} \omega_d \phi_1 \Big[(1 - \delta_0^k) H_{k,2m,k+1} (H_{k,2m+1,k+2} - H_{k,m+1,k}) \sin (k+1)\omega_d t \\
& - H_{k,2m,|k-1|} (H_{k,2m+1,k} - H_{k,2m+1,|k-1|}) \sin |k-1| \omega_d t \Big] \\
& + 2b_0 \omega_d^2 p \cos k\omega_d t \Big[\delta_0^k H^2_{k,2m,k+1} + \frac{1}{2} (H_{k,2m,k+1} - H_{k,2m,|k-1|})^2 \Big] \Bigg\}.
\end{aligned}
\tag{3.19}
$$

The effects of the carrier with constant deviation are discussed in Part I. The contribution of the signal component is illustrated in Fig. 2, where it is seen (from the ordinate scale) that the signal power varies as the square of the deviation f_d. The general determination of the noise spectrum may be made in the usual way with the aid of (2.2) and (3.5).

The example of a strong carrier is instructive. Then, with the Gaussian distribution as a particular example, we see from (2.42) and (2.47) that when there is no limiting the output spectrum is

$$
\begin{aligned}
W_0(f)_{\lambda=1} \backsimeq 2\pi^{1/2} \kappa^2 \beta^2 \omega_b b_0 [& (\Omega_d + \Omega)^2 \exp \{-(\Omega_d + \Omega)^2\} \\
& + (\Omega_d - \Omega)^2 \exp \{-(\Omega_d - \Omega)^2\}], \qquad \Omega_d \equiv \frac{\omega_d}{\omega_b},
\end{aligned}
\tag{3.20}
$$

since here the Fourier development of $[\cos (\eta_2 - \eta_1)]_{av}$. is simply $A_n \cos \omega_d t$, $A_n = \delta_n^0$. The mean output power is $(\omega_b \kappa \beta)^2 b_0/2$, which is proportional to $-\phi_2(0)$, and is independent of modulation, which in this case is the deviation of the carrier from resonance. This can be observed at once from (2.40) or directly by integration of (3.20). Figure 7 shows the spectrum (3.20) for various ratios Ω_d.

The spectrum after extreme limiting is found in a similar manner to be

$$
W_0(f)_{\lambda=2} \backsimeq \frac{32}{\pi^{3/2}} \kappa^2 \beta^2 \Big(\frac{R_0^2}{2b_0} \Big) \omega_b b_0 p^{-1} [\Omega^2 \exp \{-(\Omega_d + \Omega)^2\} + \Omega^2 \exp \{-(\Omega_d - \Omega)^2\}],
\tag{3.21}
$$

$$
p^2 \gg 1.
$$

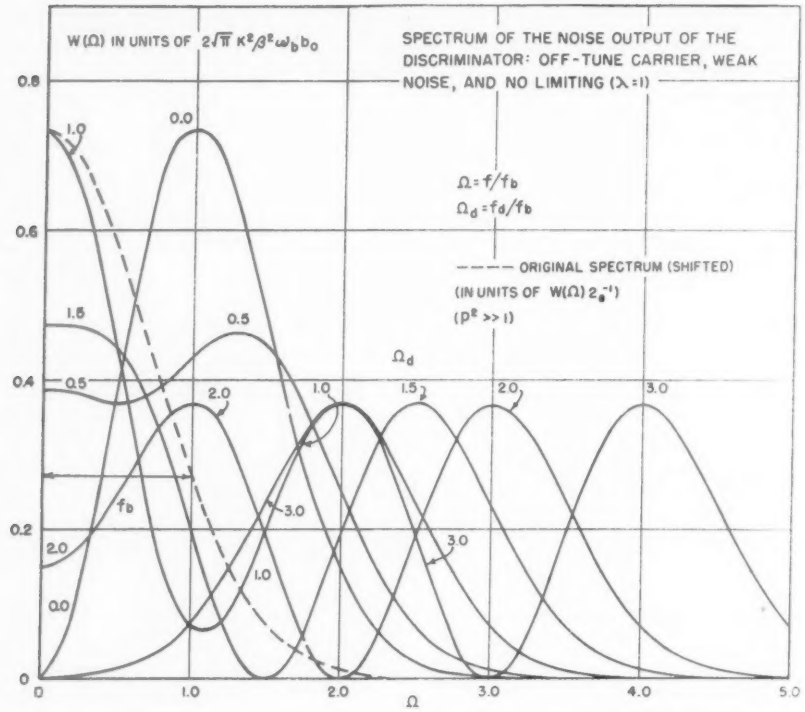

FIG. 7. Low-frequency noise spectrum following discrimination, for a weak noise and an off-tune carrier and no limiting.

Figure 8 illustrates the spectrum in this instance. The mean power output of the discriminator may be obtained from

$$(W_0)_{\lambda=2} \backsimeq \frac{32}{\pi^{3/2}}\,\kappa^2\beta^2\left(\frac{R_0^2}{2b_0}\right)\omega_b b_0 p^{-1}\int_0^\infty \Omega^2[\exp\{-(\Omega_d+\Omega)^2\}+\exp\{-(\Omega_d-\Omega)^2\}]\,d\Omega\frac{\omega_b}{2\pi} \tag{3.22}$$

$$\backsimeq \frac{16}{\pi^2}(\kappa\beta\omega_b)^2\left(\frac{R_0^2}{2b_0}\right)\frac{b_0}{p}\left(\Omega_d^2+\frac{1}{2}\right),$$

showing that the noise depends on Ω_d^2, as can be seen from Fig. 8.

The distribution in the neighborhood of $f = 0$ is easily determined with the help of Taylor's series

$$\exp\{-(\Omega_d \pm \Omega)^2\} = (2\pi)^{1/2}\sum_{l=0}^{\infty}(\pm 1)^l\frac{(2\Omega^2)^{l/2}}{l!}\phi^{(l)}(2^{1/2}\Omega_d); \tag{3.23}$$

$$\phi^{(l)} \equiv \frac{1}{(2\pi)^{1/2}}\frac{d^l}{d(2^{1/2}\Omega_d)^l}(\exp\{-\Omega_d^2\}),$$

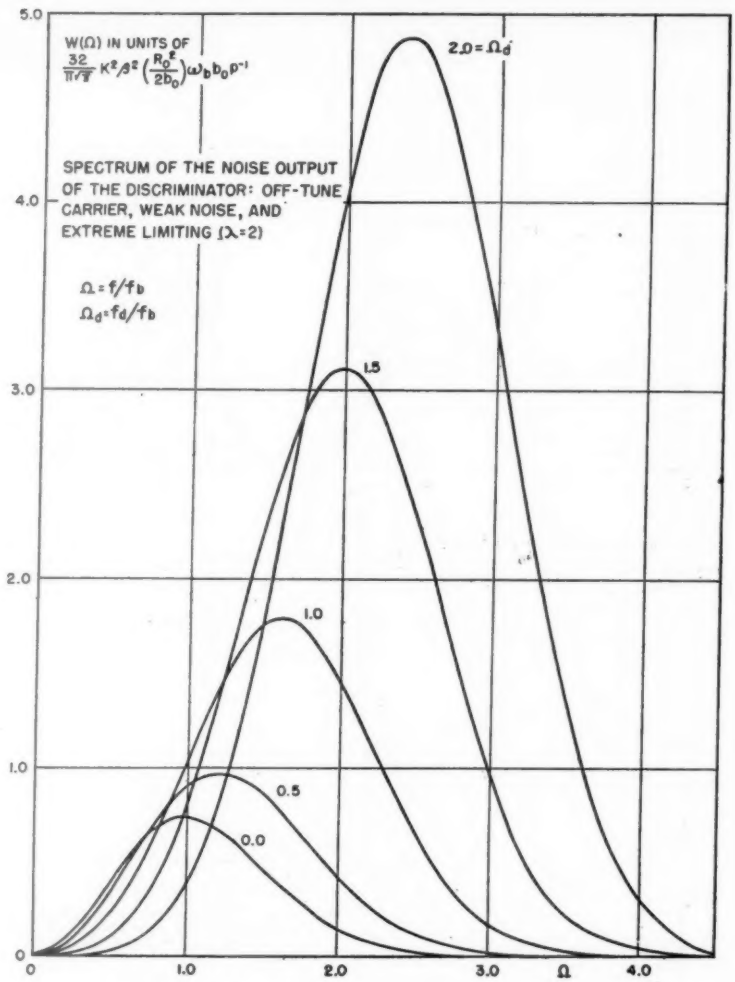

FIG. 8. The same as Fig. 7, but for extreme limiting.

a development which is valid for *all* Ω and Ω_d . Equations (3.20) and (3.21) are accordingly modified to

$$W_0(f)_{\lambda=1} \simeq 4\pi^{1/2}\kappa^2\beta^2\omega_b b_0\{\Omega_d^2 - 2\Omega\Omega_d^2 + \Omega^2(1 - 5\Omega_d^2 + 2\Omega_d^4) + \cdots\} \exp\{-\Omega_d^2\}, \tag{3.24}$$

$$\Omega^3 \ll 1$$

and

$$W_0(f)_{\lambda=2} \leq \frac{64}{\pi^{3/2}} \kappa^2\beta^2\omega_b b_0 p^{-1}\left(\frac{R_0^2}{2b_0}\right)\{\Omega^2 + \cdots\} \exp\{-\Omega_d^2\}. \tag{3.25}$$

A comparison of (3.24) and (3.25) bears out the general observation made following Eq. (2.47).

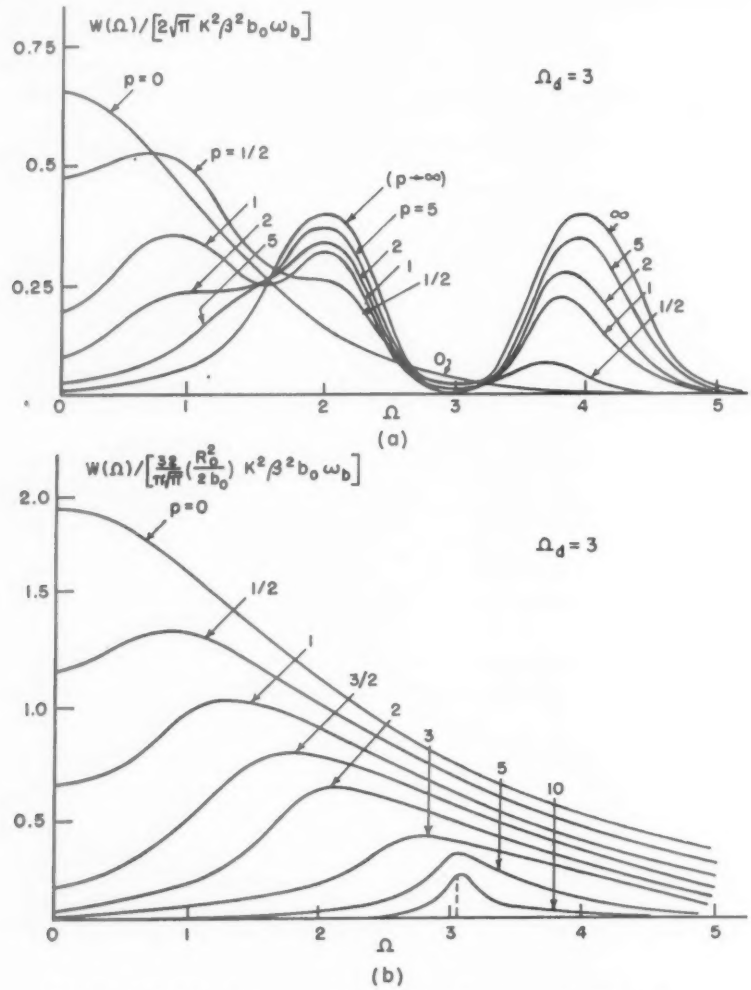

FIG. 9. Typical low-frequency spectra for an off-tune carrier: (a) no limiting, (b) extreme limiting.

4. On- and off-tune carriers modulated by a sine wave. In this case we have

$$\dot{\eta} = \omega_d + D_0 \cos \omega_a t, \quad \text{and} \quad \therefore \eta = \omega_d t + \mu \sin \omega_a t; \quad \mu \equiv \frac{D_0}{\omega_a} = \left(\frac{D_0}{\omega_b}\right) \cdot \frac{1}{\Omega_a}. \tag{3.26}$$

Here μ is the modulation index, and D_0 is the maximum deviation of the modulation, while ω_a is its angular frequency. (For such modulations the inequalities $D_0 \ll \omega_0$,

$\omega_a \ll \omega_0$ must hold.)[9] Referring to the general correlation function (2.28) we see that the modulation enters by virtue of the coefficients

$$[\cos l(\eta_2 - \eta_1)]_{av.}, \qquad [(\dot{\eta}_1 + \dot{\eta}_2) \sin l(\eta_2 - \eta_1)]_{av.},$$

and

$$[\dot{\eta}_1 \dot{\eta}_2 \cos l(\eta_2 - \eta_1)]_{av.}.$$

A straightforward application of the techniques of Appendix III allows us to write these averages as

$$[\cos l(\eta_2 - \eta_1)]_{av.} = \sum_{n=0}^{\infty} \epsilon_n J_n(l\mu)^2 \cos (n\omega_a + l\omega_d)t, \tag{3.27}$$

$$\begin{aligned}[(\dot{\eta}_1 + \dot{\eta}_2) \sin l(\eta_2 - \eta_1)]_{av.} = \sum_{n=0}^{\infty} \epsilon_n J_n(l\mu) \Big\{ & \frac{D_0}{2} J_{n-1}(l\mu) \sin [(n-1)\omega_a + l\omega_d]t \\ & + \left[\omega_d J_n(l\mu) + \frac{D_0}{2} (J_{n+1}(l\mu) + J_{n-1}(l\mu)) \right] \sin (n\omega_a + l\omega_d)t \\ & + \frac{D_0}{2} J_{n+1}(l\mu) \sin [(n+1)\omega_a + l\omega_d]t \Big\},\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}[\dot{\eta}_1 \dot{\eta}_2 \cos l(\eta_2 - \eta_1)]_{av.} = \sum_{n=0}^{\infty} \epsilon_n J_n(l\mu) \Big\{ & \left[\frac{D_0 \omega_d}{2} J_{n-1}(l\mu) + \frac{D_0^2}{4} J_n(l\mu) \right] \\ & \times \cos [(n-1)\omega_a + l\omega_d]t \\ & + \left[\omega_d^2 J_n(l\mu) + \frac{D_0 \omega_d}{2} (J_{n+1}(l\mu) + J_{n-1}(l\mu)) + \frac{D_0^2}{4} (J_{n+2}(l\mu) \right. \\ & \left. + J_{n-2}(l\mu)) \right] \cos (n\omega_a + l\omega_d)t \\ & + \left[\frac{D_0 \omega_d}{2} J_{n+1}(l\mu) + \frac{D_0^2}{4} J_n(l\mu) \right] \cos [(n+1)\omega_a + l\omega_d]t \Big\}.\end{aligned} \tag{3.29}$$

When the carrier is tuned to resonance, ω_d vanishes and (3.27) to (3.29) simplify considerably. However, a precise determination of the spectrum in either instance ($\omega_d = 0$, or $\omega_d \neq 0$) is a very tedious process. Fortunately, for many purposes the qualitative and semiquantitative conclusions obtained from the limiting behavior of the general case are sufficient; (see Part I).

We note that for strong carriers ($p^2 \gg 1$) the output spectrum may be found as before from (2.42) or (2.47). Equation (3.26) shows that the Fourier expansion of $[\cos (\eta_2 - \eta_1)]_{av.}$ is $A_n \cos (n\omega_a + \omega_d)t$, where $A_n = \epsilon_n J_n(\mu)^2$, so that for no limiting and our usual Gaussian noise distribution we find

$$\begin{aligned}W_0(f)_{\lambda=1} \simeq 2\pi^{1/2} \kappa^2 \beta^2 \omega_b b_0 \sum_{n=0}^{\infty} \epsilon_n J_n^2(\mu) \Big\{ & (n\Omega_a + \Omega_d + \Omega)^2 \exp \{-(n\Omega_a + \Omega_d + \Omega)^2\} \\ & + (n\Omega_a + \Omega_d - \Omega)^2 \exp \{-(n\Omega_a + \Omega_d - \Omega)^2\} \Big\}, \qquad \Omega_a \equiv \frac{\omega_a}{\omega_b},\end{aligned} \tag{3.30}$$

and the mean output power is simply $(\kappa^2\beta^2\omega_b^2)b_0/2$, [cf. (2.40)]. For superlimiting we obtain

$$W_0(f)_{\lambda=2} \simeq \frac{32}{\pi^{3/2}}\,\kappa^2\beta^2\omega_b b_0\left(\frac{R_0^2}{2b_0}\right)p^{-1}\Omega^2\sum_{n=0}^{\infty}\epsilon_n J_n^2(\mu)\Big\{\exp\{-(n\Omega_a+\Omega_d+\Omega)^2\} + \exp\{-(n\Omega_a+\Omega_d-\Omega)^2\}\Big\}. \tag{3.31}$$

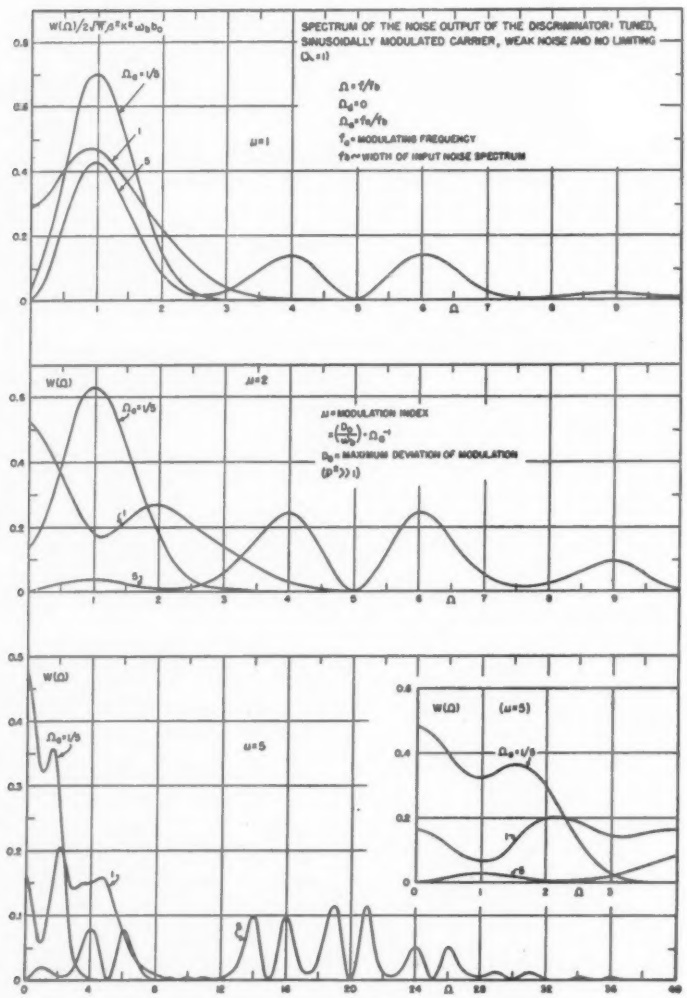

FIG. 10. Low-frequency noise spectra following discrimination, for a weak noise and a sinusoidally-modulated carrier on tune.

The mean output power associated with the noise in this instance is

$$(W_0)_{\lambda=2} = \int_0^\infty W_0(f)_{\lambda=2}\, df \simeq \frac{16}{\pi^2}(\kappa\beta\omega_b)^2 b_0 \left(\frac{R_0^2}{2b_0}\right) p^{-1}\left\{\frac{1}{2} + \Omega_d^2 + \sum_{n=0}^{\infty} \epsilon_n (n^2\Omega_a^2 + 2n\Omega_a\Omega_d) J_n^2(\mu)\right\}. \tag{3.32}$$

Figures 10 and 11 illustrate the variation in spectra for different modulation indices and modulation frequencies in the somewhat less general case of zero displacement from resonance, $\Omega_d = 0$.

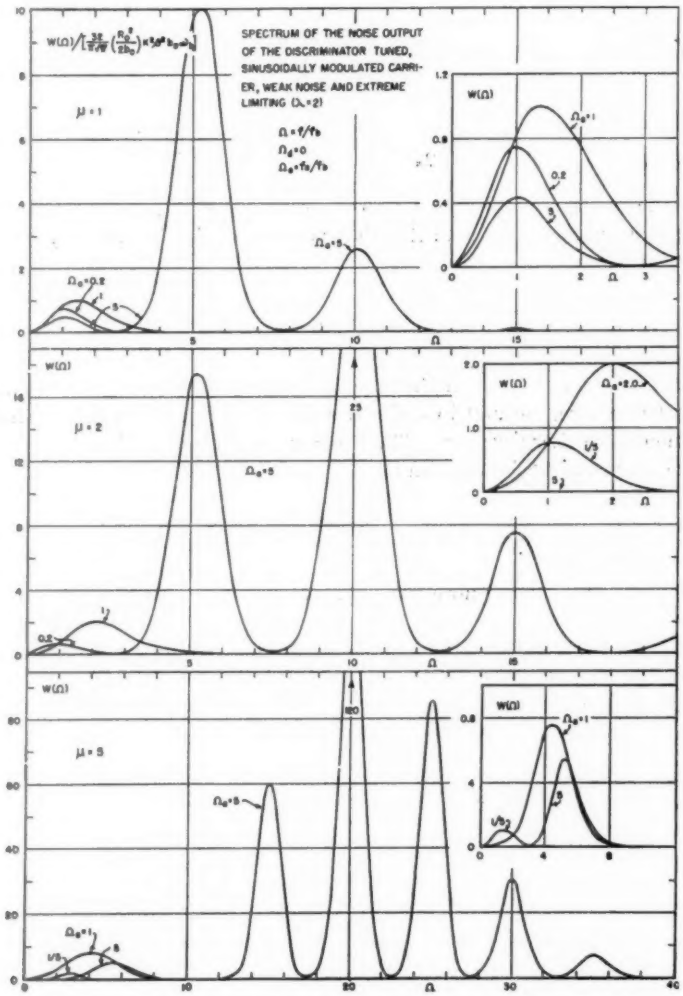

FIG. 11. The same as Fig. 10, but for extreme limiting.

APPENDIX I: Second-order probability density, its characteristic function and associated moments

Let X_k ($k = 1, 2, 3, \cdots, s$) be s random variables, each of which in turn is the sum of a large number (N) of independent random variables x_{ik} ($i = 1, 2, 3, \cdots, N$), distributed according to some law. Then the central limit theorem[27] states that the distribution of the X's is normal (i.e., multivariate Gaussian) in s dimensions, subject to certain restrictions [see e.g. S. O. Rice,[27] Eqs. (2.10-3)] on the distribution of the x_{ik}'s comprising X_k, restrictions which are assumed to be satisfied here. The probability density of the X_k's is

$$W_2(X_1, \cdots, X_s; t) = \frac{\exp\{-\tilde{\mathbf{X}}\mathbf{M}\mathbf{X}/2\}}{[(2\pi)^s \mid \mu \mid]^{1/2}} = \frac{1}{(2\pi)^{s/2} \mid \mu \mid^{1/2}} \exp\left[-\frac{1}{2}\sum_{k=1}^{s}\sum_{l=1}^{s}\frac{\mu^{kl}}{\mid \mu \mid} X_k X_l\right], \tag{A1.1}$$

where $\mathbf{M}$ is the reciprocal matrix to μ, with elements $\mathfrak{y}^{kl}/\mid \mu \mid$; μ^{kl} is the cofactor of μ_{kl}; $\mid \mu \mid$ is the determinant of μ, and $\tilde{\mathbf{X}}$ is the transpose of $\mathbf{X}$. The matrix $\mathfrak{y}$ is symmetrical and has elements

$$\mu_{kl} = \sum_{i}^{N} \overline{x_{ik}x_{il}}, \tag{A1.2}$$

in which x_{ik} is the i-th element in the sum comprising X_k and x_{il} is a similar element in X_l. We assume here, without loss of generality, that $\sum_i \overline{x_{ik}} = 0$, i.e., that $\overline{X_k} = 0$. All odd moments are therefore zero. The Fourier transform of $W_2(X_1, \cdots, X_s; t)$ is known as the *characteristic function* and is[8]

$$F_2(z_1, \cdots, z_s; t) = \exp\left[-\frac{1}{2}\sum_{k=1}^{s}\sum_{l=1}^{s}\mu_{kl}z_k z_l\right]. \tag{A1.3}$$

Now, in particular, we denote here the various "in-phase" and "out-of-phase" components $V_{c1}, \cdots, V_{s2}$ and their derivatives $\dot{V}_{c1}, \cdots, \dot{V}_{s2}$ [which by (2.6), (2.7) and the properties (2.4) are random variables] by $X_1, \cdots, X_8$, ($s = 8$). The subscript 1 on V_c, $\dot{V}_c$, or V_s, $\dot{V}_s$ refers to an initial time and the subscript 2 to a time t later. The elements of our matrix $\mathfrak{y}$ may be computed as follows, from (2.6) and (2.7) in (A1.2). For example, in the instance of μ_{11} we obtain

$$\mu_{11} = \overline{V_{c1}V_{c1}} = \left\{\sum_{n=1}^{N}(a_n \cos\omega_n' t_0 + b_n \sin\omega_n' t_0) \times \sum_{n=1}^{N}(a_m \cos\omega_m' t_0 + b_m \sin\omega_m' t_0)\right\}_{\text{stat. av.}} = \sum_{n=1}^{N}(\overline{a_n^2} + \overline{b_n^2})/2 \to \int_0^\infty w(f)\,df, \tag{A1.4}$$

with the aid of the characteristic properties (2.4) on passing to the limit $N \to \infty$. It is

[27] See S. O. Rice, Bell Syst. Tech. J. **23**, 282 (1944), sec. 2.10. D. M. I. also contains a discussion of some of these questions—Appendix I.

assumed throughout that $w(f)$ is such that the various integrals involving $w(f)$ converge. All 64 moments ($\mu^{kl} = \mu^{lk}$, $\mu_{kl} = \mu_{lk}$) are similarly found to be

$$\mu_{11} = \overline{V_{c1}^2} = \mu_{22} = \overline{V_{c2}^2} = \mu_{33} = \overline{V_{s1}^2} = \mu_{44} = \overline{V_{s2}^2} = b_0 ;$$

$$\mu_{55} = \overline{\dot{V}_{c1}^2} = \mu_{66} = \overline{\dot{V}_{c2}^2} = \mu_{77} = \overline{\dot{V}_{s1}^2} = \mu_{88} = \overline{\dot{V}_{s2}^2} = b_2 ;$$

$$\mu_{17} = \overline{V_{c1}\dot{V}_{s1}} = \mu_{28} = \overline{V_{c2}\dot{V}_{s2}} = -\mu_{35} = -\overline{V_{s1}\dot{V}_{c1}} = -\mu_{46} = -\overline{V_{s2}\dot{V}_{c2}} = b_1 ;$$

$$\mu_{13} = \overline{V_{c1}V_{s1}} = \mu_{24} = \overline{V_{c2}V_{s2}} = \mu_{15} = \overline{V_{c1}\dot{V}_{c1}}$$

$$= \mu_{37} = \overline{V_{s1}\dot{V}_{s1}} = \mu_{26} = \overline{V_{c2}\dot{V}_{c2}} = \mu_{48} = \overline{V_{s2}\dot{V}_{s2}} = 0; \tag{A1.5}$$

$$\mu_{12} = \overline{V_{c1}V_{c2}} = \mu_{34} = \overline{V_{s1}V_{s2}} = \phi_0(t);$$

$$\mu_{14} = \overline{V_{c1}V_{s2}} = -\mu_{23} = \overline{-V_{c2}V_{s1}} = \lambda_0(t);$$

$$\mu_{15} = \overline{V_{c1}\dot{V}_{c2}} = \mu_{38} = \overline{V_{s1}\dot{V}_{s2}} = -\mu_{52} = -\overline{\dot{V}_{c1}V_{c2}} = -\mu_{74} = -\overline{\dot{V}_{s1}V_{s2}} = \phi_1(t);$$

$$\mu_{18} = \overline{V_{c1}\dot{V}_{s2}} = \mu_{27} = \overline{V_{c2}\dot{V}_{s1}} = -\mu_{54} = -\overline{\dot{V}_{c1}V_{s2}} = -\mu_{63} = -\overline{\dot{V}_{c2}V_{s1}} = \lambda_1(t);$$

$$\mu_{56} = \overline{\dot{V}_{c1}\dot{V}_{c2}} = \mu_{78} = \overline{\dot{V}_{s1}\dot{V}_{s2}} = -\phi_2(t);$$

$$\mu_{58} = \overline{\dot{V}_{c1}\dot{V}_{s2}} = -\mu_{67} = -\overline{\dot{V}_{c2}\dot{V}_{s1}} = -\lambda_2(t).$$

Here we have

$$b_n = \int_0^\infty (\omega - \omega_0)^n w(f)\, df, \qquad \phi_n(t) = \frac{\partial^n}{\partial t^n}\int_0^\infty w(f)\cos(\omega - \omega_0)t\, df,$$

$$\lambda_n(t) = \frac{\partial^n}{\partial t^n}\int_0^\infty w(f)\sin(\omega - \omega_0)t\, df. \tag{A1.6}$$

When the input spectrum $w(f)$ is symmetrical, $\lambda_n(t)$ vanishes, as does also b_{2i+1} ($j = 0, 1, 2, \cdots$). The probability density (A1.1) in such cases is given by

$$W_2(V_{c1}, \cdots, \dot{V}_{s2}; t) = (2\pi)^{-4} \mid \mu \mid^{-1/2} \exp\left[-\frac{1}{2 \mid \mu \mid}\left\{\mu^{11}(V_{c1}^2 + V_{c2}^2 + V_{s1}^2 + V_{s2}^2)\right.\right.$$

$$+ \mu^{33}(\dot{V}_{c1}^2 + \dot{V}_{c2}^2 + \dot{V}_{s1}^2 + \dot{V}_{s2}^2) + 2\mu^{12}(V_{c1}V_{c2} + V_{s1}V_{s2})$$

$$+ 2\mu^{34}(\dot{V}_{c1}\dot{V}_{c2} + \dot{V}_{s1}\dot{V}_{s2}) + 2\mu^{13}(V_{c1}\dot{V}_{c1} + V_{s1}\dot{V}_{s1} - V_{c2}\dot{V}_{c2} - V_{s2}\dot{V}_{s2}) \tag{A1.7}$$

$$\left.\left. + 2\mu^{14}(V_{c1}\dot{V}_{c2} + V_{s1}\dot{V}_{s2} - V_{c2}\dot{V}_{c1} - V_{s2}\dot{V}_{s1})\right\}\right].$$

The determinant $|\mu|$ and cofactors μ^{kl} are

$$
\begin{aligned}
|\mu| &= [(b_0b_2 - \phi_1^2)^2 + (\phi_1^2 - \phi_0\phi_2)^2 - (\phi_1^4 + b_0^2\phi_2^2 + b_2^2\phi_0^2)]^2 \\
\mu^{11} &= |\mu|^{1/2}[b_0(b_2^2 - \phi_2^2) - b_2\phi_1^2], \qquad \mu^{12} = |\mu|^{1/2}[-\phi_1^2\phi_2 - \phi_0(b_2^2 - \phi_2^2)], \\
\mu^{13} &= -|\mu|^{1/2}\phi_1[\phi_0b_2 + b_0\phi_2], \qquad \mu^{14} = -|\mu|^{1/2}\phi_1[\phi_0b_2 + \phi_0\phi_2 - \phi_1^2], \\
\mu^{33} &= |\mu|^{1/2}[b_2(b_0^2 - \phi_0^2) - b_0\phi_1^2], \qquad \mu^{34} = |\mu|^{1/2}[\phi_2(b_0^2 - \phi_0^2) + \phi_1^2b_0],
\end{aligned}
\tag{A1.8}
$$

The characteristic function (A1.3) is represented by (2.13). When the spectrum $w(f)$ is not symmetrical, the resulting expressions for W_2 and F_2 are correspondingly more involved, containing $\lambda_n(t)$, $(n = 0, 1, 2)$. Specifically, the characteristic function is now

$$
\begin{aligned}
F_2(z_1, \cdots, z_8; t) = \exp\Big(-\frac{1}{2}\Big\{ & b_0(z_1^2 + z_2^2 + z_3^2 + z_4^2) + b_2(z_5^2 + z_6^2 + z_7^2 + z_8^2) \\
& + 2\phi_0(t)(z_1z_2 + z_3z_4) + 2\phi_1(t)(z_1z_6 - z_2z_5 + z_8z_3 - z_4z_7) \\
& - 2\phi_2(t)(z_5z_6 + z_7z_8) + 2\lambda_0(t)(z_1z_4 - z_2z_3) + 2\lambda_1(t)(z_1z_8 + z_2z_7 - z_3z_6 - z_4z_5) \\
& - 2\lambda_2(t)(z_5z_8 - z_6z_7)\Big\}\Big).
\end{aligned}
\tag{A1.9}
$$

APPENDIX II: Evaluation of K_1 and K_2 when $R_0 \to \infty$

The integrals K_1 and K_2 (Eq. 2.20) apparently are divergent when there is no limiting ($\mu = 1/2$ or $\lambda = 1$, and $R_0 \to \infty$), and it is not immediately evident that the results of integration (2.21) still apply in this case. The difficulty may be removed if we return to the general form (1.11) for arbitrary clipping. Our expressions for K_1 and K_2 accordingly become

$$
\begin{aligned}
K_1 = -\frac{2i}{\pi} \lim_{(R_0 \to \infty)} \int_{\mathbf{C}} \left(\frac{1 - \exp\{-iR_0\lambda\}}{\lambda^2}\right) d\lambda \iint_{-\infty}^{\infty} dx\, dy \\
\times \frac{xJ_1(\lambda(x^2 + y^2)^{1/2})}{x^2 + y^2} \exp\{-ixz - iy\xi\}
\end{aligned}
\tag{A2.1}
$$

and

$$
\begin{aligned}
K_2 = -\frac{2i}{\pi} \lim_{(R_0 \to \infty)} \int_{\mathbf{C}} \left(\frac{1 - \exp\{-iR_0\lambda\}}{\lambda^2}\right) d\lambda \iint_{-\infty}^{\infty} dx\, dy \\
\times \frac{yJ_1(\lambda(x^2 + y^2)^{1/2})}{x^2 + y^2} \exp\{-ixz - iy\xi\}.
\end{aligned}
\tag{A2.2}
$$

Transforming to polar coordinates gives us for K_1

$$K_1 = -\frac{2i}{\pi}\lim_{(R_0\to\infty)}\int_C\left(\frac{1-\exp\{-iR_0\lambda\}}{\lambda^2}\right)d\lambda\int_0^\infty dr\,J_1(\lambda r) \times\int_0^{2\pi}\cos\theta\exp\{iZr\cos(\theta-\psi)\}\,d\theta, \tag{A2.3}$$

where

$$\cos\psi=\frac{z}{(z^2+\xi^2)^{1/2}},\qquad \sin\psi=\frac{\xi}{(z^2+\xi^2)^{1/2}},\qquad \text{and}\qquad Z=(z^2+\xi^2)^{1/2}\geq 0.$$

We employ next Eq. (1.4), so that (A2.3) is found to be

$$K_1=-\frac{4z}{Z}\lim_{(R_0\to\infty)}\int_C\left(\frac{1-\exp\{-iR_0\lambda\}}{\lambda^2}\right)\left[\int_0^\infty J_1(r\lambda)J_1(Zr)\,dr\right]d\lambda. \tag{A2.4}$$

The integral in r is a special case of Weber and Schafheitlin's integral[10] and has the values

$$\left.\begin{aligned}\int_0^\infty J_1(r\lambda)J_1(rZ)\,dr&=\frac{\lambda}{2Z^2}\cdot{}_2F_1(1/2,\,3/2;\,2;\,\lambda^2/Z^2),\qquad Z\geq\lambda\geq 0\\&=\frac{Z}{2\lambda^2}\cdot{}_2F_1(1/2,\,3/2;\,2;\,Z^2/\lambda^2),\qquad \lambda\geq Z\geq 0\end{aligned}\right\}. \tag{A2.5}$$

At $Z=\lambda$ the functions are identical, viz., $2\Gamma(0)/\pi$, but logarithmically divergent. Since this divergence is only logarithmic, the integral in λ is still convergent, including the point $Z=\lambda$, and this independently of R_0. We observe now that (A2.5) may be extended to negative values of λ if we replace λ by $|\lambda|$ and multiply the expressions (A2.5) by (-1). The integrands containing the real part of $(1-\exp[-iR_0\lambda])$ then cancel, and we are left with

$$K_1=-\frac{4zi}{Z}\lim_{(R_0\to\infty)}\left[\frac{1}{Z^2}\int_0^Z\frac{\sin R_0\lambda}{\lambda}\,{}_2F_1(1/2,\,3/2;\,2;\,\lambda^2/Z^2)\,d\lambda + Z\int_Z^\infty\frac{\sin R_0\lambda}{\lambda}\,{}_2F_1(1/2,\,3/2;\,2;\,Z^2/\lambda^2)\,d\lambda\right]. \tag{A2.6}$$

Utilizing the fact that $\lim_{(R_0\to\infty)}(\sin R_0\lambda)/\lambda$ is a delta-function,[6,8] with the value $\pi\delta(\lambda-0)$, we have finally

$$K_1=-\frac{2\pi iz}{Z^3}=-\frac{2\pi iz}{(z^2+\xi^2)^{3/2}}; \tag{A2.7}$$

the second integral in (A2.6) vanishes because of the property $\int_a^b f(x)\delta(x-0)\,dx=0$, $b>a>0$. A corresponding expression for K_2 is obtained on replacing z by ξ in the numerator of (A2.7). If we set $\mu=1/2$ in (2.21) we see that the result is precisely that given above, cf. (A2.7).

APPENDIX III: Integration for $\mathbf{R}(t)_N$

The Bessel expansions for the trigonometric exponentials in (2.27) are

$$\exp\{-\phi_0\rho_1\rho_2\cos(\theta_2-\theta_1)+iA_0\rho_1\cos(\theta_1-\eta_1)+iA_0\rho_2\cos(\theta_2-\eta_2)\}$$

$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}(-1)^k\epsilon_k I_k(\phi_0\rho_1\rho_2)i^{l+n}\epsilon_l\epsilon_n J_l(A_0\rho_1)J_n(A_0\rho_2) \tag{A3.1}$$

$$\cos k(\theta_2-\theta_1)\cos l(\theta_1-\eta_1)\cos n(\theta_2-\eta_2).$$

The integration over θ_1 and θ_2 indicated in (2.27) is accomplished with the help of

$$\frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix}\sin\psi k\cdot\sin\psi m\\ \cos\psi k\cdot\cos\psi m\end{pmatrix}d\psi=\delta_m^k/\epsilon_m \quad\text{and}\quad \frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix}\sin\psi k\cdot\cos\psi m\\ \cos\psi k\cdot\sin\psi m\end{pmatrix}d\psi=0. \tag{A3.2}$$

We at length obtain the following five expressions for the average over θ_1 and θ_2 :

$$C_{1,ln}^{(k)}=\frac{\cos(n\eta_2-l\eta_1)}{2\epsilon_l\epsilon_n}[\delta_l^{k+1}\delta_n^{k+1}+\delta_l^{|k-1|}\delta_n^{|k-1|}], \tag{A3.3}$$

$$C_{2,ln}^{(k)}=\frac{\cos(n\eta_2-l\eta_1)}{2\epsilon_l\epsilon_n}\left[\delta_l^k\delta_n^k-\frac{1}{2}\delta_l^{k+2}\delta_n^{k+2}-\frac{1}{2}\delta_l^{|k-2|}\delta_n^{|k-2|}\right], \tag{A3.4}$$

$$\begin{aligned}C_{3,ln}^{(k)}=\frac{1}{4\epsilon_l\epsilon_n}[&\delta_n^{k+1}(\delta_l^{k+2}+\delta_l^k)\cos\eta_1\sin(n\eta_2-l\eta_1)\\&+\delta_n^{k+1}(\delta_l^{k+2}-\delta_l^k)\sin\eta_1\cos(n\eta_2-l\eta_1)-\delta_n^{k-1}\delta_l^{|k-1|+1}\sin n\eta_2\cos(l-1)\eta_1\\&+\delta_n^{|k-1|}\delta_l^k\cos n\eta_2\sin(l-1)\eta_1-\delta_n^{k-1}\sin n\eta_2(\delta_l^{|k-1|-1}\cos\eta_1\cos l\eta_1\\&-\delta_l^{|k-1|-1}\sin\eta_1\sin l\eta_1)+\delta_n^{|k-1|}\cos n\eta_2(\delta_l^{k-2}\cos\eta_1\sin l\eta_1+\delta_l^{|k-2|}\sin\eta_1\cos l\eta_1)],\end{aligned} \tag{A3.5}$$

$$\begin{aligned}C_{4,ln}^{(k)}=\frac{1}{4\epsilon_l\epsilon_n}[&\delta_l^{k+1}(\delta_n^{k+2}+\delta_n^k)\cos\eta_2\sin(n\eta_2-l\eta_1)\\&-\delta_l^{k+1}(\delta_n^{k+2}-\delta_n^k)\sin\eta_2\cos(n\eta_2-l\eta_1)\\&+\delta_l^{k-1}\delta_n^{|k-1|+1}\sin l\eta_1\cos(n-1)\eta_2-\delta_l^{|k-1|}\delta_n^k\cos l\eta_1\sin(n-1)\eta_2\\&+\delta_l^{k-1}\sin l\eta_1(\delta_n^{|k-1|-1}\cos\eta_2\cos n\eta_2-\delta_n^{|k-1|-1}\sin\eta_2\sin n\eta_2)\\&-\delta_l^{|k-1|}\cos l\eta_1(\delta_n^{k-2}\cos\eta_2\sin n\eta_2+\delta_n^{|k-2|}\sin\eta_2\cos n\eta_2)],\end{aligned} \tag{A3.6}$$

and

$$
\begin{aligned}
C^{(k)}_{5,ln} = \frac{1}{4\epsilon_n\epsilon_l}\,[&\cos\eta_2\cos n\eta_2\cos\eta_1\cos l\eta_1(\delta_n^{k+1}+\delta_n^{|k-1|})(\delta_l^{k+1}+\delta_l^{|k-1|})\\
&+\cos\eta_2\cos n\eta_2\sin\eta_1\sin l\eta_1(\delta_n^{k+1}+\delta_n^{|k-1|})(\delta_l^{k+1}-\delta_l^{k-1})\\
&+\sin\eta_2\sin n\eta_2\cos\eta_1\cos l\eta_1(\delta_n^{k+1}-\delta_n^{k-1})(\delta_l^{k+1}+\delta_l^{|k-1|})\\
&+\sin\eta_2\sin n\eta_2\sin\eta_1\sin l\eta_1(\delta_n^{k+1}-\delta_n^{k-1})(\delta_l^{k+1}-\delta_l^{k-1})\\
&+\cos\eta_2\sin n\eta_2\cos\eta_1\sin l\eta_1(\delta_n^{k+1}+\delta_n^{k-1})(\delta_l^{k+1}+\delta_l^{k-1})\\
&+\cos\eta_2\sin n\eta_2\sin\eta_1\cos l\eta_1(\delta_n^{k+1}+\delta_n^{k-1})(\delta_l^{|k-1|}-\delta_l^{k+1})\\
&+\sin\eta_2\cos n\eta_2\cos\eta_1\sin l\eta_1(\delta_n^{|k-1|}-\delta_n^{k+1})(\delta_l^{k+1}+\delta_l^{k-1})\\
&+\sin\eta_2\cos n\eta_2\sin\eta_1\cos l\eta_1(\delta_n^{|k-1|}-\delta_n^{k+1})(\delta_l^{|k-1|}-\delta_l^{k+1})]
\end{aligned}
\tag{A3.7}
$$

with the convention that $\delta_b^{-a} = -\delta_b^a$. The correlation function may now be written

$$
\begin{aligned}
R_0(t)_N = \gamma_\lambda^2 \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\epsilon_k\epsilon_l\epsilon_n(-1)^{k+(l+n)/2}\int_0^\infty \rho_1^{\lambda-2}J_l(\rho_1A_0)\exp(-b_0\rho_1^2/2)\,d\rho_1\\
\times\int_0^\infty \rho_2^{\lambda-2}J_n(\rho_2A_0)\exp(-b_0\rho_2^2/2)I_k(\phi_0\rho_1\rho_2)\,d\rho_2\\
\times\{\phi_2C^{(k)}_{1,ln}+\phi_1^2\rho_1\rho_2C^{(k)}_{2,ln}+iA_0\phi_1\dot{\eta}_1\rho_1C^{(k)}_{3,ln}+iA_0\phi_1\dot{\eta}_2\rho_2C^{(k)}_{4,ln}-A_0^2\dot{\eta}_1\dot{\eta}_2C^{(k)}_{5,ln}\}.
\end{aligned}
\tag{A3.8}
$$

The series in n and l are eliminated because of the Kronecker deltas in (l, k) and (n, k), which appear everywhere in pairs in the various C's. If we separate the variables ρ_1 and ρ_2 by expanding $I_k(\phi_0\rho_1\rho_2)$, integrate, and reduce the C's, we finally obtain (2.28).

APPENDIX IV: Some properties of $H_{k,2m+n,q}(p;\lambda)$

The determination of spectra requires a wide range of values of the order $(k, 2m + n, q)$ of the amplitude functions $H_{k,2m+n,q}(p;\lambda)$. These values are best found with the help of recurrence relations, which greatly facilitate the work of computation. The amplitude function H is described by (Eq. 2.29)

$$
\begin{aligned}
H_{k,2m+n,q}(p;\lambda) = \frac{\Gamma[(k+2m+n+q+\lambda-1)/2]p^{q/2}}{q!b_0^{(n+\lambda-1)/2}2^{(3-\lambda)/2}[m!(m+k)!]^{1/2}}\\
\times {}_1F_1([k+2m+n+q+\lambda-1]/2;\,q+1;\,-p),
\end{aligned}
\tag{A4.1}
$$

$k, m, n, q \geq 0$ and integral. Here q is equal to k plus an integer, and n is either zero or unity. Using this fact we distinguish, for convenience, two classes of functions, denoted

respectively by superscript zero and superscript unity. The amplitude functions are thus only of double order (in k and m).

Recurrence relations between the H's are readily found from those of the confluent hypergeometric function ${}_1F_1$, a table of which we list below,[28] after multiplication by the appropriate factors [cf. (A4.1)]. We have

	$F_{\alpha+}$	$F_{\alpha-}$	$F_{\beta-}$	$F_{\beta-}$	F
1.	α	$\alpha - \beta$			$\beta - 2\alpha \mp p$
2.	$\alpha\beta$		$\pm p(\beta - \alpha)$		$-\beta(\alpha \pm p)$
3.	α			$1 - \beta$	$(\beta - \alpha - 1)$
4.		$-\beta$	$\mp p$		β
5.		$\alpha - \beta$		$\beta - 1$	$1 - \alpha \mp p$
6.			$\pm p(\beta - \alpha)$	$\beta(\beta - 1)$	$\beta(1 - \beta \mp p)$

(A4.2)

The subscripts on F heading the columns refer to an addition or subtraction of unity in α or β of $F = {}_1F_1(\alpha; \beta; \pm p)$, the rows list the factors which multiply the heading F's, and the upper sign wherever $(\pm)$ occurs refers to ${}_1F_1(\alpha; \beta; +p)$, while the lower refers to ${}_1F_1(\alpha; \beta; -p)$. The sum of each row is zero.

There are five distinct H's, corresponding to $q = k + 1, \mid k - 1 \mid$, with $n = 0$, and $q = k, k + 2, \mid k - 2 \mid$, with $n = 1$. For each amplitude function, two recurrence formulas are needed to cover the complete manifold (k, m): one for k fixed, $m = 0, 1, \cdots$, the other for m constant, $k = 0, 1, 2, \cdots$. Since m appears only in α of ${}_1F_1(\alpha; \beta; -p)$ [see (A4.1) above] the first row of (A4.2) may be used directly in the former case, viz:

$$[(\beta - \alpha){}_1F_1(\alpha - 1; \beta; -p) + (2\alpha - \beta - p){}_1F_1(\alpha; \beta; -p)]/\alpha$$

$$= {}_1F_1(\alpha + 1; \beta; -p). \qquad \text{(A4.3)}$$

In the latter, where k is varied, both α and β contain k additively, and so we seek a relation between F, $F_{\alpha-\beta-}$, and $F_{\alpha+\beta+}$. Now if we change β to $\beta - 1$ in 4. of (A4.2), and again alter α to $\alpha + 1$, use 5. and eliminate $F_{\beta-}$ and $F_{\alpha+}$ from these three relations, we obtain

$$\frac{\beta}{p\alpha}\{-(\beta - 1){}_1F_1(\alpha - 1; \beta - 1; -p) + (\beta - 1 + p){}_1F_1(\alpha; \beta; -p)\} \qquad \text{(A4.4)}$$

$$= {}_1F_1(\alpha + 1; \beta + 1; -p).$$

[28] See Appendix 4 in ref. 7.

The general recurrence formulas follow from (A4.3) and (A4.4), and are

$$\frac{1}{[(m+2)(m+2+k)]^{1/2}}\left\{-\left(\frac{k+2m+n+\lambda-1-q}{2}\right)\right.$$
$$\times\left(\frac{k+2m+n+q+\lambda-1}{2}\right)\frac{H^{(n)}_{k,2m+n,q}}{[(m+1)(m+k+1)]^{1/2}} \tag{A4.5}$$
$$\left.+(2m+n+k+\lambda-p)H^{(n)}_{k,2m+n+2,q}\right\}=H^{(n)}_{k,2m+n+4,q}$$

and

$$\frac{1}{(m+k+2)^{1/2}}\left\{-\left(\frac{k+2m+n+q+\lambda-1}{2}\right)\frac{H^{(n)}_{k,2m+n,q}}{(m+k+1)^{1/2}}\right.$$
$$\left.+\frac{(q+p+1)}{p^{1/2}}H^{(n)}_{k+1,2m+n,q+1}\right\} \tag{A4.6}$$
$$=H^{(n)}_{k+2,2m+n,q+2}\,.$$

We observe from (A4.1) that $\alpha=(k+2m+n+q+\lambda-1)/2$ is either a positive integer or half-integer. When α is a half-integer we use [29]

$${}_1F_1(\mu+1/2;2\mu+1;\pm p)=\frac{2^{2\mu}\Gamma(\mu+1)}{(\pm p)^{\mu}}\exp\{\pm p/2\}I_{\mu}(\pm p/2). \tag{A4.7}$$

This is a convenient form for computation because of the available tables of $e^{-x}I_0(x)$ and $e^{-x}I_1(x)$.[21] When α is integral, ${}_1F_1$ may be expressed in closed form as a polynomial in p and e^{-p}. In general, four functions must be known initially for $n=0$ or $n=1$, in order that (A4.5) and (A4.6) can be used to give us a complete set of H's.

The initial functions [30] to be used in the recurrence relations when $n=0$ $(q=k+1)$ to give us $H^{(0)}_{k,2m,k+1}$ are

$$H^{(0)}_{001}(p;\lambda)=a_\lambda p^{1/2}{}_1F_1(\lambda/2;2;-p)=a_1p^{1/2}e^{-p/2}[I_0(p/2)+I_1(p/2)], \quad (\lambda=1) \tag{A4.8a}$$
$$=a_2p^{-1/2}(1-e^{-p}) \quad , \quad (\lambda=2) \tag{A4.8b}$$

$$H^{(0)}_{021}=\frac{\lambda a_\lambda}{2}p^{1/2}{}_1F_1(\lambda/2+1;2;-p)=a_1p^{1/2}e^{-p/2}[I_0(p/2)-I_1(p/2)]/2 \tag{A4.9a}$$
$$=a_2p^{1/2}e^{-p}, \tag{A4.9b}$$

$$H^{(0)}_{102}=\frac{\lambda a_\lambda}{4}p\,{}_1F_1(\lambda/2+1;3;-p)=a_1e^{-p/2}I_1(p/2) \tag{A4.10a}$$
$$=a_2(1-e^{-p}-pe^{-p})/p \tag{A4.10b}$$

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[29] Ref. 21, pp. 191.

[30] W. R. Bennett, J. Acoust. Soc. Amer. 15, 164 (1944).

and

$$H^{(0)}_{122} = (\lambda/2+1)\frac{\lambda a_\lambda}{4\cdot 2^{1/2}}\, p\ {}_1F_1(\lambda/2+2;3;-p)$$

$$= a_1 e^{-p/2}[pI_0(p/2) - (p+1)I_1(p/2)]/2^{3/2} = \frac{a_\lambda}{c_\lambda 2^{1/2}} H^{(1)}_{032} \quad \text{(A4.11a)}$$

$$= a_2 p e^{-p}/2^{1/2} \quad \text{(A4.11b)}$$

where $a_\lambda = \Gamma(\lambda/2)/b_0^{(\lambda-1)}2^{(3-\lambda)/2}$;

$$\text{in particular, } a_1 = \pi^{1/2}/2 \text{ and } a_2 = (2b_0)^{-1/2}. \quad \text{(A4.12)}$$

A similar technique is applied to obtain $H^{(0)}_{k,2m,|k-1|}$ $(n = 0,\ q = |k-1|)$. When $k = 0$, $H^{(0)}_{0,2m,1}$ may be obtained from (A4.8-A4.11) above. The initial functions here are

$$H^{(0)}_{100} = a_\lambda\ {}_1F_1(\lambda/2;1;-p) = a_1 e^{-p/2} I_0(p/2) \quad \text{(A4.13a)}$$

$$= a_2 e^{-p}, \quad \text{(A4.13b)}$$

$$H^{(0)}_{120} = \frac{\lambda a_\lambda}{2^{3/2}}\ {}_1F_1(\lambda/2+1;1;-p) = a_1 e^{-p/2}[(1-p)I_0(p/2) + pI_1(p/2)]/2^{3/2} \quad \text{(A4.14a)}$$

$$= a_2(1-p)e^{-p}/2^{1/2}, \quad \text{(A4.14b)}$$

$$H^{(0)}_{201} = \frac{\lambda a_\lambda}{2^{3/2}}\, p^{1/2}\ {}_1F_1(\lambda/2+1;2;-p) = a_1 p^{1/2} e^{-p/2}[I_0(p/2) - I_1(p/2)]/2^{3/2} \quad \text{(A4.15a)}$$

$$= a_2 p^{1/2} e^{-p}/2^{1/2}, \quad \text{(A4.15b)}$$

and

$$H^{(0)}_{221} = (\lambda/2+1)\frac{\lambda a_\lambda}{2\cdot 6^{1/2}}\, p^{1/2}\ {}_1F_1(\lambda/2+2;2;-p)$$

$$= a_1 \frac{p^{1/2}e^{-p/2}}{2\cdot 6^{1/2}}[(3/2-p)I_0(p/2) + (p-1/2)I_1(p/2)] = \frac{a_\lambda}{c_\lambda} H^{(1)}_{311} \quad \text{(A4.16a)}$$

$$= 2a_2 p^{1/2} e^{-p}(1-p/2)/6^{1/2} \quad \text{(A4.16b)}$$

We continue in the same way to obtain $H^{(1)}_{k,2m+1,k}$ $(n = 1,\ q = k)$. The initial functions are

$$H^{(1)}_{010} = c_\lambda\ {}_1F_1(\lambda/2;1;-p) = c_1 e^{-p/2} I_0(p/2) \quad \text{(A4.17a)}$$

$$= c_2 e^{-p}, \quad \text{(A4.17b)}$$

$$H^{(1)}_{030} = \frac{\lambda c_\lambda}{2}\ {}_1F_1(\lambda/2+1;1;-p) = \frac{c_1 e^{-p/2}}{2}[(1-p)I_0(p/2) + pI_1(p/2)] \quad \text{(A4.18a)}$$

$$= c_2(1-p)e^{-p}, \quad \text{(A4.18b)}$$

$$H^{(1)}_{111} = \lambda/2\, c_\lambda p^{1/2}{}_1F_1(\lambda/2+1;2;-p) = c_1 \frac{p^{1/2}e^{-p/2}}{2}[I_0(p/2) - I_1(p/2)] \tag{A4.19a}$$

$$= c_2 p^{1/2} e^{-p}, \tag{A4.19b}$$

and

$$H^{(1)}_{131} = (\lambda/2+1)\frac{\lambda c_\lambda}{2^{3/2}} p^{1/2}{}_1F_1(\lambda/2+2;2;-p)$$

$$= \frac{p^{1/2}c_1 e^{-p/2}}{2^{3/2}}[(3/2-p)I_0(p/2) + (p-1/2)I_1(p/2)] = 3^{1/2}H^{(1)}_{311} \tag{A4.20a}$$

$$= 2c_2 p^{1/2}e^{-p}(1-p/2)/2^{1/2}, \tag{A4.20b}$$

and here

$$c_\lambda = \Gamma(\lambda/2)/b_0^{\lambda/2}2^{(2-\lambda)/2}; \text{ specifically, } c_1 = (\pi/2b_0)^{1/2} \text{ and } c_2 = b_0^{-1}. \tag{A4.21}$$

There remain $H^{(1)}_{k,2m+1,k+2}$ $(n=1, q=k+2)$ and $H^{(1)}_{k,2m+1,|k-2|}$, $(n=1, q=|k-2|)$. For the former we find the initial functions to be

$$H^{(1)}_{012} = (\lambda/4)\, c_\lambda p \cdot {}_1F_1(\lambda/2+1;3;-p) = c_1 e^{-p/2} I_1(p/2) \tag{A4.22a}$$

$$= c_2(1 - e^{-p} - pe^{-p})/p, \tag{A4.22b}$$

$$H^{(1)}_{032} = (\lambda/2+1)\frac{\lambda c_\lambda}{4} p \cdot {}_1F_1(\lambda/2+2;3;-p) = c_1 \frac{e^{-p/2}}{2}[pI_0(p/2) - (p+1)I_1(p/2)] \tag{A4.23a}$$

$$= c_2 p e^{-p}, \tag{A4.23b}$$

$$H^{(1)}_{113} = (\lambda/2+1)\frac{\lambda c_\lambda}{12} p^{3/2}{}_1F_1(\lambda/2+2;4;-p) = \frac{c_1 e^{-p/2}}{2p^{1/2}}[(4+p)I_1(p/2) - pI_0(p/2)] \tag{A4.24a}$$

$$= c_2 p^{-3/2}(2 - 2e^{-p} - 2pe^{-p} - p^2e^{-p}), \tag{A4.24b}$$

and

$$H^{(1)}_{133} = \frac{(2+\lambda/2)(1+\lambda/2)\lambda p^{3/2}c_\lambda}{3\cdot 2^{5/2}}\, {}_1F_1(\lambda/2+3;4;-p)$$

$$= \frac{c_1}{2^{5/2}} p^{-1/2}e^{-p/2}[(2p^2+p)I_0(p/2) - (2p^2+3p+4)I_1(p/2)] \tag{A4.25a}$$

$$= c_2 p^{3/2}e^{-p}/2^{1/2}. \tag{A4.25b}$$

We have finally, for $H^{(1)}_{k,2m+1,|k-2|}$,

$$H^{(1)}_{210} = \frac{\lambda c_\lambda}{2^{3/2}}\, {}_1F_1(\lambda/2+1;1;-p) = \frac{c_1 e^{-p/2}}{2^{3/2}}[(1-p)I_0(p/2) + pI_1(p/2)] \tag{A4.26a}$$

$$= \frac{c_2}{2^{1/2}}(1-p)e^{-p}, \tag{A4.26b}$$

Spectrum	*Analytic Form* ($x = \omega_b t$)	$r_0(t)$	$r_1(t)$	$r_2(t)$	$\Delta(t) \equiv r_1^2 - r_0 r_2$
(1.) *Gaussian*	$w(f) = W_0 \exp\{-(\omega - \omega_0)^2/\omega_b^2\}$	$\exp(-x^2/4)$	$-\omega_b \frac{x}{2} \exp(-x^2/4)$	$\frac{\omega_b^2}{2}\left(\frac{x^2}{2} - 1\right) \cdot \exp(-x^2/4)$	$\frac{\omega_b^2}{2} \exp(-x^2/2)$
(2.) *Rectangular*	$w(f) = W_0$, $f_0 - f_b < f < f_0 + f_b$ $= 0$, $f_0 - f_b > f$ $f_0 + f_b < f$	$\frac{\sin x}{x}$	$-\omega_b\left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right)$	$\omega_b^2\left\{2\frac{\sin x}{x^3} - 2\frac{\cos x}{x^2} - \frac{\sin x}{x}\right\}$	$\omega_b^2\left(\frac{x^2 - \sin^2 x}{x^4}\right)$
(3.) *"Optical"*	$w(f) = \frac{W_0}{1 + [(\omega - \omega_0)/\omega_b]^2}$	$\exp(-\lvert x \rvert)$	$-\omega_b \exp(-\lvert x \rvert)$, $x > 0$ 0, $x = 0$	$\omega_b^2 \exp(-\lvert x \rvert)$, $x > 0$ $-\omega_b^2 \delta(x - 0)$, $x = 0$	0, $x > 0$ $\omega_b^2 \delta(x - 0)$, $x = 0$

$$H_{230}^{(1)} = (\lambda/2 + 1)\frac{\lambda c_\lambda}{2\cdot 6^{1/2}}\,{}_1F_1(\lambda/2 + 2; 1; -p) = \frac{c_1 e^{-p/2}}{2\cdot 6^{1/2}}[(p^2 - 3p + 3/2)I_0(p/2) + (2p - p^2)I_1(p/2)] \tag{A4.27a}$$

$$= \frac{c_2 e^{-p}}{6^{1/2}}(2 - 4p + p^2), \tag{A4.27b}$$

$$H_{311}^{(1)} = (\lambda/2 + 1)\frac{\lambda c_\lambda p^{1/2}}{2\cdot 6^{1/2}}\,{}_1F_1(\lambda/2 + 2; 2; -p)$$

$$= \frac{c_1 p^{1/2}}{2\cdot 6^{1/2}} e^{-p/2}[(3/2 - p)I_0(p/2) + (p - 1/2)I_1(p/2)] \tag{A4.28a}$$

$$= 2c_2 p^{1/2} e^{-p}(1 - p/2), \tag{A4.28b}$$

and

$$H_{331}^{(1)} = (\lambda/2 + 2)(\lambda/2 + 1)\frac{\lambda c_\lambda}{4\cdot 6^{1/2}}\,p^{1/2}\,{}_1F_1(\lambda/2 + 3; 2; -p)$$

$$= \frac{c_1 p^{1/2} e^{-p/2}}{4\cdot 6^{1/2}}[(p^2 - 9p/2 + 15/4)I_0(p/2) - (p^2 - 7p/2 + 3/4)I_1(p/2)] \tag{A4.29a}$$

$$= \frac{c_2 p^{1/2} e^{-p}}{2\cdot 6^{1/2}}(p^2 - 6p + 6). \tag{A4.29b}$$

APPENDIX V: Evaluation of Series (Noise alone); Remarks on Optical and Rectangular Spectra

We note here some possible methods of summing the series (3.7) and (3.9) in the instance of noise alone. For no limiting ($\lambda = 1$) Kummer's transformation[31] puts the series (3.7) in a form that is more rapidly convergent. Here we write

$A_m = 2(m + 1)\exp\{\beta/(m + 1)\}$; $a_m = (1/2)_m^2 \exp\{-\beta/(m + 1)\}/(m!)^2(m + 1)^{3/2}$;

$$2\beta = \omega^2/\omega_b^2\,,$$

and

$$\lambda_m = A_m - A_{m+1}\frac{a_{m+1}}{a_m}$$

$$= 2(m + 1)\exp\{\beta/(m + 1)\}\left[1 - \left(\frac{m + 1}{m + 2}\right)^{1/2}\frac{(m + 1/2)^2}{(m + 1)^2}\right].$$

The series becomes

$$\sum_{m=0}^{\infty} a_m = A_0 a_0 + \sum_{m=0}^{\infty}(1 - \lambda_m)a_m = 2 + \sum_{m=0}^{\infty}(1 - \lambda_m)a_m\,. \tag{A5.1}$$

[31] E. P. Adams, *Smithsonian mathematical formulas and tables of elliptic functions*, Publication No. 2672, Smithsonian Institution, 1939.

Because $\lim_{m\to\infty} \lambda_m \to 1$, the series $\sum_m (1 - \lambda_m)a_m$ converges more rapidly than $\sum_m a_m$ alone.

On the other hand, for superlimiting ($\lambda = 2$), we may use Euler's summation formula[32] to sum the remainder after $m + 1 = M$ terms. Our series (3.9) becomes

$$\sum_{m=0}^{\infty} \frac{\exp\{-\beta/(m+1)\}}{(m+1)^{3/2}} = \sum_{m=0}^{M-1} \frac{\exp\{-\beta/(m+1)\}}{(m+1)^{3/2}} + (\pi/\beta)^{1/2}\,\Theta([\beta/M]^{1/2}) + M^{-3/2}\exp\{-\beta/M\}\left\{\left[\frac{1}{2} + \frac{1}{8M} - \frac{\beta}{12M^2}\right] + \frac{1}{720M^3}\left[-\frac{105}{8} + \frac{105\beta}{4M} - \frac{21\beta^2}{2M^2} + \frac{\beta^3}{M^3}\right] + \cdots\right\} \tag{A5.2}$$

The function $\Theta(x) = 2\pi^{-1/2}\int_0^x \exp(-y^2)\,dy$ is the error-function.[24] Observe that when $\beta \to \infty$ the series approaches the value $(\pi/\beta)^{1/2}$.

* * * * * *

When there is only random noise entering the limiter-discriminator elements, the correlation function of the output [cf. (3.2) or (3.8)] in the two extremes of no limiting and superlimiting is proportional to $\Delta(t) \equiv [r_1^2(t) - r_0(t)r_2(t)]$, where r_0 , r_1 , r_2 are defined by (2.14) and (2.15). It is instructive to determine $\Delta(t)$ for the case of "optical" and rectangular spectra, as well as for the Gaussian distribution used extensively in the present paper. The results appear in the table. Here $\delta(t - 0)$ is the familiar delta-function of Dirac, and the spectral "width-factors" ω_b for the different distributions are related by

$$(\omega_b)_{\text{Gauss}} = \frac{2}{\pi^{1/2}}(\omega_b)_{\text{rect.}} = \pi^{1/2}(\omega_b)_{\text{opt.}}\,, \tag{A5.3}$$

if the input noise power (b_0) and the maximum spectral intensity W_0 are the same in these cases.

When the input spectrum is "optical," the output noise spectrum is found from (2.2) and (3.1) to be

$$W(f)_{\text{opt.}} = \gamma_\lambda^2 b_0^{2-\lambda} 2^{\lambda-1}\Gamma(\lambda/2)^2\,\frac{\omega_b}{\pi}\int_0^\infty \cos\omega t\;{}_2F_1(\lambda/2,\,\lambda/2;\,2;\,r_0^2)\,\delta(t-0)\,dt \tag{A5.4}$$

$$= \gamma_\lambda^2 b_0^{2-\lambda} 2^{\lambda-1}\,\frac{\omega_b^2}{2\pi}\,\frac{\Gamma(\lambda/2)^2\Gamma(2-\lambda)}{\Gamma(2-\lambda/2)^2}, \qquad (f > 0).$$

There is no *d-c* component, since $\Delta(\infty) = 0$. The spectrum has a uniform intensity for *all* frequencies, and hence the power output is infinite. This catastrophe is never realized physically, because it is not possible to produce an exact optical distribution. Owing to parasitic effects in the circuit elements the falling-off of $w(f)$ at the higher and lower frequencies, away from resonance, is always greater than the ideal optical behavior, and so $\Delta(t)$ is finite for all values of t. The output spectrum, widely spread when parasitic effects are minimal, nevertheless encloses a finite area and hence finite power. The saturation of the discriminator in practical cases imposes a further deviation from the optical shape.

[32]Reference 31, Section 1.86. We follow a suggestion of Rice[4] here.

CURRENTS ON THE SURFACE OF AN INFINITE CYLINDER EXCITED BY AN AXIAL SLOT*

BY

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1. Summary. The distribution of surface-current density is determined on an axially infinite, perfectly conducting cylinder. A constant distribution of electric field across a narrow axial slot is assumed given.

2. Introduction. The analysis of the electromagnetic field radiated by a perfectly conducting circular cylinder of finite length fed by an axial slot becomes quite involved

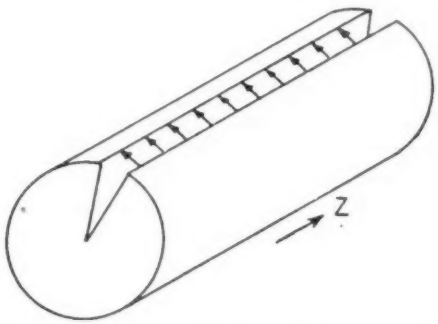

FIG. A. Infinite cylinder with axial slot. Constant E across mouth of slot.

especially when the end surfaces are rigorously treated. However, if the slot is situated half way between the ends of the cylinder, the peripheral surface current distribution and the radiation pattern in the plane through the slot and perpendicular to the axis of the cylinder is approximately the same as for an infinite cylinder with an infinitely long axial slot. Therefore, in so far as this peripheral surface current and radiation pattern is concerned, it is only necessary to solve the following problem.

The problem investigated is the distribution of currents on the surface of an axially infinite cylinder of radius a fed from an axial slot across which is maintained a constant distribution of electric field defined by

$$E = \theta_1 E_\theta(a, \theta, Z) = \theta_1 E_0 G(\theta) e^{-i\omega t}, \tag{1}$$

where $G(\theta)$ is defined as follows:

$$G(\theta) = \begin{cases} 1 \text{ for } -\theta_0/2 < \theta < \theta_0/2 \\ 0 \text{ for } |\theta| > \theta_0/2 \end{cases}$$

and θ_1 is the unit vector in the θ-direction of the cylindrical co-ordinates r, θ, z.

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Since the cylinder has infinite conductivity, all the components of the E-field for $r = a$ and $|\theta| > \theta_0/2$ are zero except the radial component E_r. Therefore, for $r = a$, E_θ

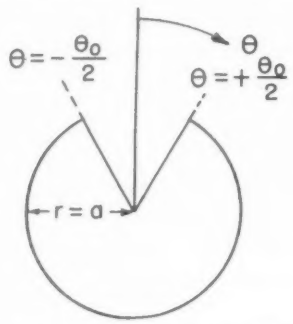


FIG. B. Cross-section of cylinder with slot.

is constant in the gap and zero outside it. The applied field, E_θ, maintains currents in the θ-direction. The field outside the cylinder has only the components E_θ, E_r, and H_z so that it is TM (transverse magnetic) and may be regarded as a superposition of cylindrical waves traveling radially outward from the surface of the cylinder. At great distances from the cylinder, E_r is negligible compared to E_θ, so that the field is predominantly TEM (transverse electromagnetic). As r approaches a, E_θ of the radiated field must approach the E_θ of the applied field, and H_z becomes numerically equal to the θ-component of the surface-current density J_θ.

3. Fourier expansion of the applied field. To obtain an analytical expression for the applied field as given by (1), it is expanded in a complex Fourier series; that is,

$$E_0 G(\theta)\, e^{-i\omega t} = \sum_{n=-\infty}^{n=+\infty} a_n\, e^{in\theta - i\omega t} \tag{2}$$

where a_n are the Fourier coefficients given by

$$a_n = E_0 (n\pi)^{-1} \sin\,(n\theta_0/2) \tag{3}$$

By substituting (3) in (2), the Fourier expansion for the applied field is

$$E_\theta = E_0 G(\theta)\, e^{-i\omega t} = E_0 \pi^{-1} \sum_{n=-\infty}^{n=+\infty} n^{-1} \sin\,(n\theta_0/2)\, e^{in\theta - i\omega t} \tag{4}$$

This infinite series represents E_θ for all values of θ between $-\pi$ and π.

4. General expression for field surrounding cylinder. The most general expressions for E_θ and H_z for $r \geq a$ are the following:[1,2]

$$E_\theta = \frac{-i\mu\omega}{k} \sum_{n=-\infty}^{n=+\infty} H_n^{(1)\prime}(kr) b_n\, e^{in\theta - i\omega t} \tag{5}$$

and

$$H_z = \sum_{n=-\infty}^{n=+\infty} H_n^{(1)}(kr) b_n\, e^{in\theta - i\omega t} \tag{6}$$

where $k = 2\pi/\lambda$, λ being the free space wavelength, and where $H_n^{(1)}(kr)$ are the Hankel functions of the first kind and order n, $H_n^{(1)\prime}(kr)$ are the derivatives $[d/d(kr)]H_n^{(1)}(kr)$,

and b_n are constant coefficients. E_r is not involved in the determination of the unknown coefficients. Expressions (5) and (6) hold for all values $r \geq a$ and for $-\pi \leq \theta \leq \pi$.

5. Matching at surface of cylinder. At the surface of the cylinder, $r = a$, expressions (4) and (5) must be identically equal for all values of θ between $-\pi$ and π. Therefore,

$$E_0(n\pi)^{-1} \sin (n\theta_0/2) = -(i\mu\omega/k)H_n^{(1)\prime}(ka)b_n . \tag{7}$$

This equation determines b_n for all values of n. Explicitly,

$$b_n = \{ikE_0 \sin (n\theta_0/2)\}/\{\mu\omega\pi n H_n^{(1)\prime}(ka)\}. \tag{8}$$

By substituting (8) into (6), H_z becomes

$$H_z = (ikE_0/\mu\omega\pi) \sum_{n=-\infty}^{n=+\infty} n^{-1} \sin (n\theta_0/2) \frac{H_n^{(1)}(ka)}{H_n^{(1)\prime}(ka)} e^{in\theta - i\omega t} \tag{9}$$

at the surface of the cylinder.

6. Surface-current density. Since J_θ is numerically equal to H_z, the surface-current density around the periphery of the cylinder is

$$J_\theta = (ikE_0/\mu\omega\pi) \sum_{n=-\infty}^{n=+\infty} n^{-1} \sin (n\theta_0/2) \frac{H_n^{(1)}(ka)}{H_n^{(1)\prime}(ka)} e^{in\theta - i\omega t} \tag{10}$$

The limits of summation in (10) may be changed by using the formulas

$$\begin{aligned} H_{-n}^{(1)}(x) &= e^{n\pi i} H_n^{(1)}(x) \\ H_{-n}^{(1)\prime}(x) &= e^{n\pi i} H_n^{(1)\prime}(x) \end{aligned} \tag{11}$$

Substituting (11) in (10), noting that $k/(\mu\omega) = 1/(120\pi)$ mhos, the following expression is obtained for J_θ.

$$J_\theta = \frac{E_0 e^{-i\omega t}}{120\pi^2} \left[\frac{i\theta_0}{2} \frac{H_0^{(1)}(ka)}{H_0^{(1)\prime}(ka)} + 2i \sum_{n=1}^{\infty} \frac{\sin (n\theta_0/2)}{n} \frac{H_n^{(1)}(ka)}{H_n^{(1)\prime}(ka)} \cos n\theta \right] \tag{12}$$

Let α be the real part and β the imaginary part of the bracketed expression in (12) so that this becomes

$$J_\theta = (120\pi^2)^{-1} E_0 e^{-i\omega t}(\alpha + i\beta) \tag{13}$$

For any given values of the parameters θ_0 and ka, α and β are functions of the angle θ. The absolute value of J_θ is given by

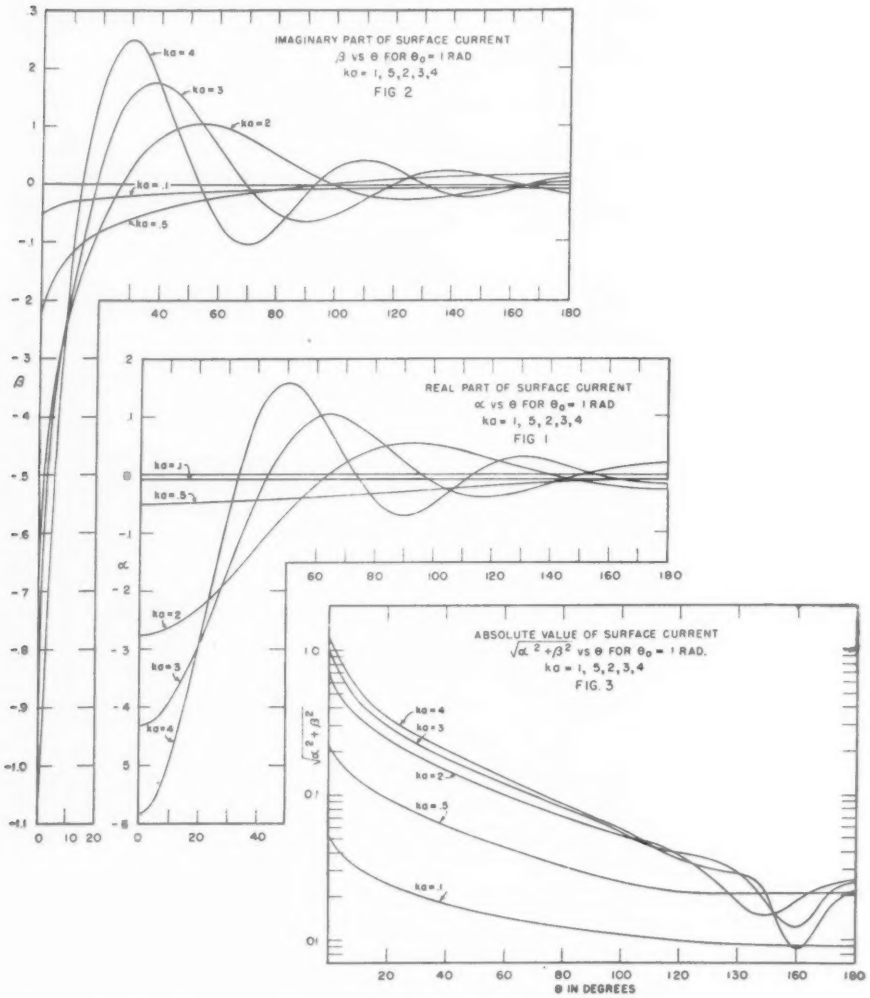
$$| J_\theta | = (120\pi^2)^{-1} E_0 (\alpha^2 + \beta^2)^{1/2} \tag{14}$$

7. Radiation pattern. The radiation pattern is the distribution of $| E_\theta |$ about the cylinder along a circle whose radius r is so large that it is in the far zone defined by $kr \gg 1$. The formula for E_θ is obtained by substituting (8) in (5). The result, which is valid for $kr \geq ka$, is

$$E_\theta = \pi^{-1} E_0 e^{-i\omega t} \sum_{n=-\infty}^{n=+\infty} \frac{\sin (n\theta_0/2)}{n} \frac{H_n^{(1)\prime}(kr)}{H_n^{(1)\prime}(ka)} e^{in\theta} \tag{15}$$

Since in (15) a very good approximation to the infinite series is obtained by summing from $n = -M$ to $n = +M$ where $M = 40$, it is permissible to use the following asymptotic form when $kr \gg 1$ and $kr \gg M$.

$$H_n^{(1)\prime}(kr) \cong i(2/\pi kr)^{1/2} e^{i(kr-[(2n+1)/4]\pi)} \tag{16}$$

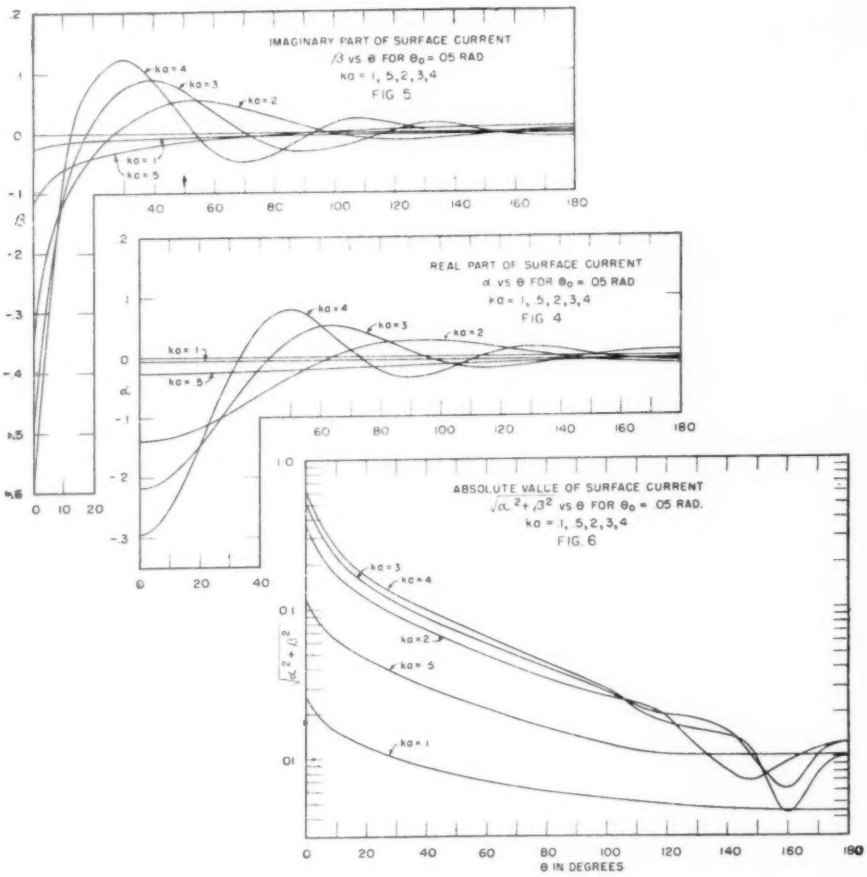

IMAGINARY PART OF SURFACE CURRENT
β VS θ FOR θ0 = 1 RAD
ka = 1, 5, 2, 3, 4
FIG 2
REAL PART OF SURFACE CURRENT
α VS θ FOR θ0 = 1 RAD
ka = 1, 5, 2, 3, 4
FIG 1
ABSOLUTE VALUE OF SURFACE CURRENT
√(α² + β²) VS θ FOR θ0 = 1 RAD.
ka = 1, 5, 2, 3, 4
FIG. 3
ka = 4
ka = 3
ka = 2
ka = .5
ka = .1
ka = 1
θ IN DEGREES

Substituting (16) into (15) and changing the limits of summation, E_θ (for far zone) becomes

$$E_\theta = \pi^{-1} E_0 e^{-i\omega t}(2/\pi kr)^{1/2} e^{ikr} e^{i/(\pi/4)} [\theta_0 \{2H_0^{(1)\prime}(ka)\}^{-1} + 2\sum_{n=1}^{n=M} \sin(n\theta_0/2) e^{-in\pi/2} \{nH_n^{(1)\prime}(ka)\}^{-1} \cos n\theta] \tag{17}$$

The factor in front of the first square bracket is a complex constant independent of θ. Denoting the complex quantity in the square brackets by $\delta + i\epsilon$, and taking the absolute value of both sides of (17), the following expression is obtained for E_θ which defines the radiation pattern

$$E_\theta \sim |\delta + i\epsilon|; \qquad |E_\theta| \sim (\delta^2 + \epsilon^2)^{1/2} \tag{18}$$

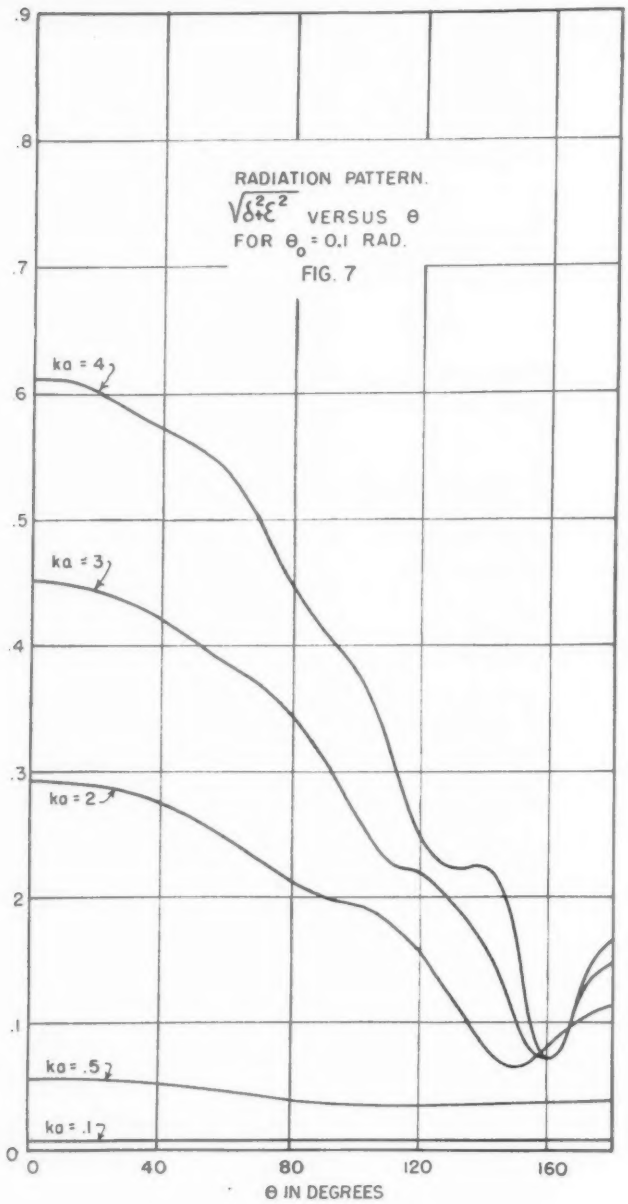

IMAGINARY PART OF SURFACE CURRENT
β vs θ FOR θ0 = 05 RAD
ka = 1, 5, 2, 3, 4
FIG 5
REAL PART OF SURFACE CURRENT
α vs θ FOR θ0 = 05 RAD
ka = 1, 5, 2, 3, 4
FIG 4
ABSOLUTE VALUE OF SURFACE CURRENT
√α² + β² vs θ FOR θ0 = 05 RAD.
ka = 1, 5, 2, 3, 4
FIG. 6
ka = 1
ka = 5
ka = 2
ka = 3
ka = 4
θ IN DEGREES

8. **Computations.** For suitably chosen values of θ_0 and ka, α and β were computed using tables of Hankel functions [3] for orders $n = 0$ to $n = 20$. For n greater than 20, use was made of the asymptotic form

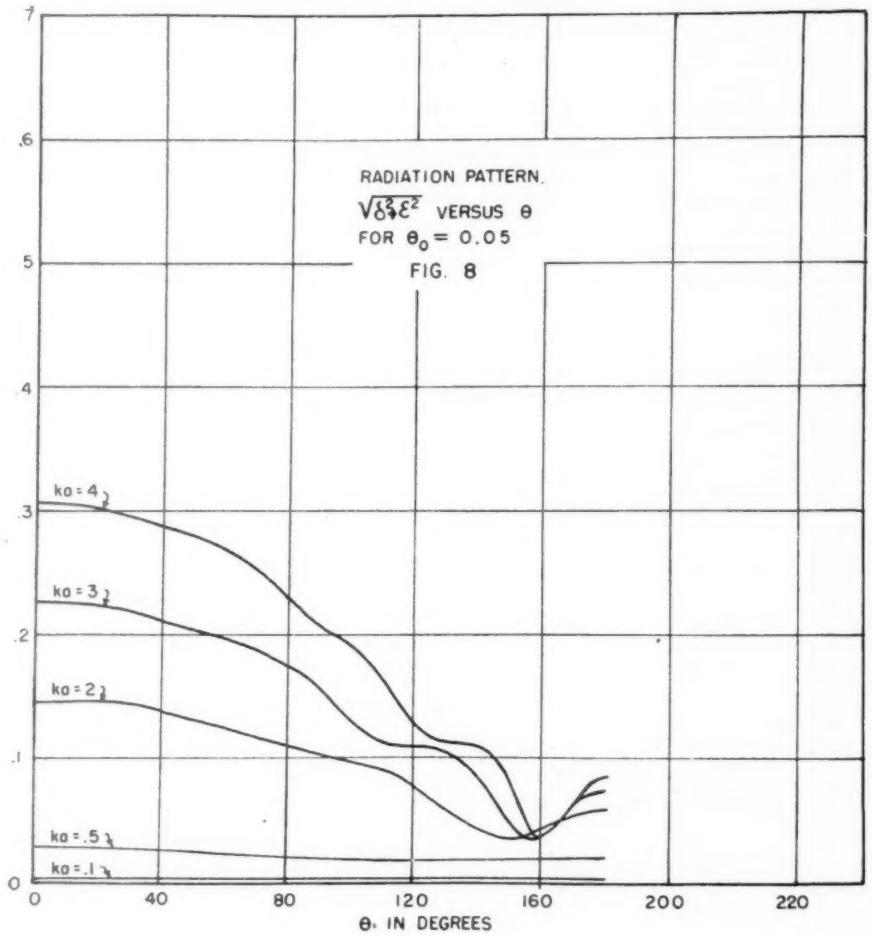
$$\frac{H_n^{(1)}(ka)}{H_n^{(1)\prime}(ka)} \to -\left(\frac{ka}{n}\right), \tag{19}$$

which holds for $(ka)^2 \ll 2n + 1$. Forty terms were used in the calculation.

9. **Results.** The current density on the periphery of cylinders of various sizes ($ka = 1/10, 1/2, 2, 3$, and 4) and for two sizes of gap ($\theta_0 = 1/10$ and $1/20$) were calculated


RADIATION PATTERN.
$\sqrt{\delta^2+\epsilon^2}$ VERSUS θ
FOR θ_0 = 0.1 RAD.
FIG. 7
ka = 4
ka = 3
ka = 2
ka = .5
ka = .1
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
0
40
80
120
160
θ IN DEGREES

using (13). The results are in Figs. 1-6. Figures 1, 2 and 3 show plots of α, β and $(\alpha^2 + \beta^2)^{1/2}$ as functions of the angle θ for $\theta_0 = 1/10$. Note that α and β are proportional, respectively, to the real and imaginary parts of the surface-current density J_θ; $(\alpha^2 + \beta^2)^{1/2}$ is proportional to the magnitude $| J_\theta |$. Figures 4, 5, and 6 show corresponding plots of α, β, and $(\alpha^2 + \beta^2)^{1/2}$ for $\theta_0 = 1/20$.



For $ka = 1/10$, the current density about the periphery of the cylinder is almost constant. For $ka = 1/2$, 2, 3, and 4 there is a standing-wave pattern about the periphery. In all the cases considered, the current near the gap is larger than the current near $\theta = 180°$. By narrowing the gap from $\theta = 1/10$ to $\theta_0 = 1/20$, the overall level of current density was decreased, since reducing the size of the gap is essentially tantamount to decreasing the strength of the source.

The radiation patterns for the two gap sizes $\theta_0 = 0.1$ and $\theta_0 = 0.05$ (Figs. 7 and 8) show that the radius of the cylinder has a marked effect. For $ka = 0.1$, the pattern is uniform. With increasing values of ka the patterns become nonuniform. This is to be expected since when ka is small the current distribution has a uniform amplitude about the periphery of the cylinder, whereas when ka is increased, standing waves obtain.

10. Acknowledgment. The authors thank Dr. C. T. Tai of Cruft Laboratory for his aid in the preparation of this paper. The computations were made by Mrs. R. Stokey.

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THE EQUIVALENT LOADING METHOD AND THE EQUIVALENT BEAM METHOD[1]

BY

PEI-PING CHEN

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Introduction. Even under the usual engineering assumptions, the analysis of beams with non-uniform cross section subjected to distributed lateral loads, axial forces, and elastic support action is often difficult. It is the purpose of this paper to present a technique by which one can reduce more complicated beam problems to standard ones. This technique consists of two successive methods which we call the equivalent loading method and the equivalent beam method.

In the *equivalent loading method* an arbitrary distribution of lateral loading on a *portion* of a beam is replaced by concentrated loads and couples so chosen that they are entirely equivalent to the given loading *so far as the remainder of the beam is concerned*. Thus a simple equivalent loading of concentrated forces and couples may be used to evaluate rigorously the stress and deflection that the given loading over a *portion* of a beam produces in the *remainder* of the beam.

In an analogous fashion, the *equivalent beam method* is used to replace a portion of a beam by an equivalent beam portion in such a way that the stresses and deflections in the remainder of the beam are not affected by the change. In this method the original portion may be of non-uniform cross section and may have axial forces and elastic support, but must be free from lateral load; the equivalent beam portion can always be chosen to consist of segments having uniform cross sections, free from axial forces and elastic restraint.

It is assumed everywhere in the paper that the axial forces are such that elastic instability does not occur.

PART I: THE EQUIVALENT LOADING METHOD

Structurally balanced loading systems

1. Fundamental fixed-ended beam problem. The equivalent loading method, which will be introduced in this paper to analyze complicated beam problems, depends on the analysis of the fundamental problem of the bending of a fixed-ended beam. The reason for this will be shown in Sec. 7.

Let us consider a fixed-ended beam of non-uniform cross section subjected to arbitrary distribution of both lateral load and axial force (see Fig. 1); furthermore, for generality, we will assume the beam to be resting on an elastic foundation. The deflection function $y(x)$ is governed by the differential equation,

$$L[y] = -\{EI(x)y''\}'' - \{P(x)y'\}' - k(x)y = F(x) \tag{1.1}$$

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together with the boundary conditions

$$y(0) = y(l) = y'(0) = y'(l) = 0 \tag{1.2}$$

where $EI(x)$ is the rigidity, always positive and finite,
$P(x)$ is the axial compression,
$k(x)$ is the foundation reaction intensity due to unit deflection ($k(x) \geqq 0$, if the beam is elastically supported),
and $F(x)$ is the lateral load distribution (continuous or discontinuous).

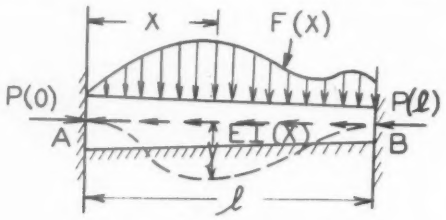

FIG. 1.

Here, as in all of the following discussion, the magnitude of the axial force is assumed to be such that buckling will not occur. The known functions, $P(x)$ and $EI(x)$, are also assumed to be piecewise differentiable and twice differentiable, respectively, in the open interval $(0, l)$.

In the derivation of Eq. (1.1), the following well-known formulas

$$M(x) = -EI(x)y''(x), \qquad S(x) = M'(x) - P(x)y'(x) \tag{1.3}$$

are used for the bending moments and shear forces. The sign convention used is shown in Fig. 2.

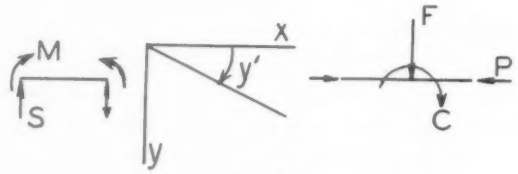

FIG. 2.

The solution of Eq. (1.1) together with the boundary conditions (1.2) may be expressed in terms of Green's function $G(x, \xi)$:

$$y(x) = \int_0^l G(x, \xi)F(\xi)\, d\xi. \tag{1.4}$$

Here $G(x, \xi)$ is the Green's function for the operator $L[y]$ with the boundary condition (1.2); stated in physical terms, it is the deflection at x due to unit lateral load at ξ, with the effects of both axial force as well as the elastic foundation taken into account (see Fig. 3).

2. Structurally balanced loading system.

Definition. A loading system $f(x)$ is said to be *structurally balanced* with respect to a fixed-ended beam AB, if and only if it produces no reaction forces and couples at

both ends. It follows from Eqs. (1.2) and (1.3) that these reaction forces and couples vanish if and only if

$$y''(0) = y''(l) = y'''(0) = y'''(l) = 0, \tag{2.1}$$

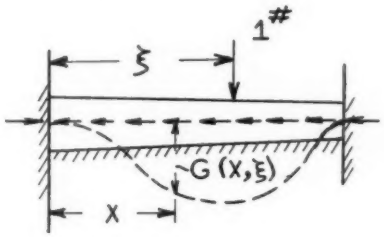

FIG. 3.

whence, from Eq. (1.4),

$$\begin{aligned} &\int_0^l G_{xx}(0, \xi)f(\xi)\, d\xi = 0, \\ &\int_0^l G_{xx}(l, \xi)f(\xi)\, d\xi = 0, \\ &\int_0^l G_{xxx}(0, \xi)f(\xi)\, d\xi = 0, \\ \text{and} \quad &\int_0^l G_{xxx}(l, \xi)f(\xi)\, d\xi = 0. \end{aligned} \tag{2.2}$$

It is shown in Sec. 4 that the functions

$$G_{xx}(0, \xi), \quad G_{xx}(l, \xi), \quad G_{xxx}(0, \xi), \quad G_{xxx}(l, \xi) \tag{2.3}$$

must be linearly independent.

The above equations (2.2) constitute necessary and sufficient conditions for structurally balanced loading systems. These conditions might be further simplified by the aid of a linear transformation (see Example I).

EXAMPLE I: To find conditions which must be satisfied by all structurally balanced loading systems acting on a fixed-ended uniform beam under the following conditions:

$$EI(x) = \text{constant}, \qquad P(x) = k(x) = 0. \tag{2.4}$$

SOLUTION: The Green's function is found to be:

$$G(x, \xi) = \begin{cases} x^2(l - \xi)^2(3\xi l - xl + 2x\xi)/6EIl^3, & x < \xi \\ \xi^2(l - x)^2(3xl - \xi l + 2\xi x)/6EIl^3, & \xi < x \end{cases} \tag{2.5}$$

from which we have

$$\begin{aligned} G_{rr}(0, \xi) &= \xi(l - \xi)^2/EIl^2, \\ G_{rr}(l, \xi) &= \xi^2(l - \xi)/EIl^2, \\ G_{rrr}(0, \xi) &= (l - \xi)^2(2\xi - l)/EIl^3, \end{aligned} \tag{2.6}$$

and $$G_{rrr}(l, \xi) = \xi^2(3l + 2\xi)/EIl^3.$$

The above set of functions may be further simplified to the following set of linearly independent functions,

$$1, \quad \xi, \quad \xi^2, \quad \xi^3 \tag{2.7}$$

by a linear transformation

$$\xi = EIG_{rr}(0, \xi) + \frac{7}{5} EIG_{rr}(l, \xi) + \frac{l}{5} EIG_{rrr}(l, \xi), \qquad \text{etc.}$$

Thus, according to Eq. (2.2), necessary and sufficient conditions for a structurally balanced loading system are the following equations:

$$\int_0^l \xi^m f(\xi)\, d\xi = 0, \qquad m = 0, 1, 2, 3. \tag{2.8}$$

3. Auxiliary functions of a beam.

Definition. Four functions $\varphi_1(\xi)$, $\varphi_2(\xi)$, $\varphi_3(\xi)$, $\varphi_4(\xi)$ are said to form a set of *auxiliary functions* of a fixed-ended beam if and only if they are linearly independent and are such that the relations

$$\int_0^l \varphi_i(\xi) f(\xi)\, d\xi = 0, \qquad i = 1, 2, 3, 4 \tag{3.1}$$

hold for every loading system $f(x)$ which is structurally balanced with respect to that beam AB. For example, the functions (2.3) form a set of auxiliary functions.

In order to find the differential equation for auxiliary functions, we rewrite Eq. (1.1) in a more general form:

$$L[y] = \alpha(x)y^{iv} + \beta(x)y''' + \gamma(x)y'' + \delta(x)y' + \epsilon(x)y = F(x). \tag{3.2}$$

For any auxiliary function $\varphi(x)$ and any arbitrary structurally balanced loading system $f(x)$, we have

$$\begin{aligned} 0 &= \int_0^l \varphi(\xi) f(\xi)\, d\xi \\ &= \int_0^l \varphi(\xi)\{\alpha(\xi)y^{iv} + \beta(\xi)y''' + \gamma(\xi)y'' + \delta(\xi)y' + \epsilon(\xi)y\}\, d\xi \\ &= \{Ay''' + By'' + Cy' + Dy\}\Big|_{\xi=0}^{\xi=l} + \int_0^l y(\xi)L^*[\varphi(\xi)]\, d\xi \end{aligned}$$

by integration by parts where $A = \alpha(\xi)\varphi(\xi)$, $B = -\{\alpha(\xi)\varphi(\xi)\}' + \beta(\xi)\varphi(\xi)$; etc., and $L^*[\varphi] = \{\alpha(\xi)\varphi\}^{iv} - \{\beta(\xi)\varphi\}''' + \{\gamma(\xi)\varphi\}'' - \{\delta(\xi)\varphi\}' + \epsilon(\xi)\varphi$. This last operator L^* is usually called the *adjoint operator* of L. Since $y = y' = y'' = y''' = 0$ at both ends (see Eqs. (1.2) and (2.1)), we get

$$\int_0^l y(\xi)L^*[\varphi(\xi)]\,d\xi = 0 \tag{3.3}$$

for which a sufficient condition is

$$L^*[\varphi(\xi)] = 0. \tag{3.4}$$

To show that Eq. (3.4) is also necessary for an auxiliary function $\varphi(\xi)$, we assume that $L^*[\varphi(\xi)] > 0$ at some point ξ. By continuity, there exists a neighborhood of ξ,

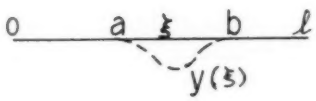

FIG. 4.

say (a, b), such that $L^*[\varphi(\xi)] > 0$ for ξ in (a, b). Let us construct a deflection function $y(x)$ defined by (see Fig. (4)):

$$\begin{cases} y = 0 & \text{when} \quad x \text{ is not in } (a, b) \\ y > 0 & \text{when} \quad x \text{ is in } (a, b) \end{cases} \tag{3.5}$$

so that Eqs. (1.2) and (2.1) are satisfied, but not Eq. (3.3). Then there exists a structurally balanced loading system $F(x)$ obtainable from Eq. (3.2) and Eq. (3.5) and an auxiliary function $\varphi(\xi)$ such that

$$\int_0^l \varphi(\xi)F(\xi)\,d\xi = \int_0^l y(\xi)L^*[\varphi(\xi)]\,d\xi \neq 0$$

which contradicts Eq. (3.3). Hence the necessity of Eq. (3.4) is proved.

Since the operator L defined by Eq. (1.1) is self-adjoint, any set of linearly independent complementary functions[2] of that equation may be used as a set of auxiliary functions.

4. Necessary and sufficient conditions for structurally balanced loading systems in terms of auxiliary functions. Necessary and sufficient conditions for structurally balanced loading systems have already been obtained in terms of Green's function (see Eq. (2.2)). Now we try to express them in terms of auxiliary functions.

From the definition of auxiliary functions, it is seen that

$$\int_0^l \varphi_i(\xi)f(\xi)\,d\xi = 0, \qquad i = 1, 2, 3, 4 \tag{4.1}$$

are necessary conditions for a structurally balanced loading system $f(x)$. Here the $\varphi_i(\xi)$ are the linearly independent complementary functions of Eq. (1.1).

[2] E. L. Ince, *Ordinary differential equations*, Dover Publications, New York, 1944, p. 402.

To prove the sufficiency of Eq. (4.1), that is, to show that a loading system $f(x)$ is structurally balanced if it satisfies Eq. (4.1), we need to have the following additional structural stability condition:

$$\Delta[\varphi] = \begin{vmatrix} \varphi_1(0) & \varphi_1(l) & \varphi_1'(0) & \varphi_1'(l) \\ \varphi_2(0) & \varphi_2(l) & \varphi_2'(0) & \varphi_2'(l) \\ \varphi_3(0) & \varphi_3(l) & \varphi_3'(0) & \varphi_3'(l) \\ \varphi_4(0) & \varphi_4(l) & \varphi_4'(0) & \varphi_4'(l) \end{vmatrix} \neq 0. \tag{4.2}$$

For if any loading system $f(x)$ satisfies the conditions (4.1), then by repeating the the argument of Sec. 3, we obtain

$$\{A_i(\xi)y'''(\xi) + B_i(\xi)y''(\xi)\}\Big|_{\xi=0}^{\xi=l} = 0, \qquad i = 1, 2, 3, 4 \tag{4.3}$$

due to the facts

(i) that $y(0) = y(l) = y'(0) = y'(l) = 0$ for a fixed-ended beam,

and (ii) that $L^*[\varphi_i(\xi)] = 0$ obtained from the given condition (4.1).

Since the determinant of the coefficients of $y'''(0)$, $y'''(l)$, $y''(0)$ and $y''(l)$ in Eq. (4.3),

$$\begin{vmatrix} A_1(0) & A_1(l) & B_1(0) & B_1(l) \\ A_2(0) & A_2(l) & B_2(0) & B_2(l) \\ A_3(0) & A_3(l) & B_3(0) & B_3(l) \\ A_4(0) & A_4(l) & B_4(0) & B_4(l) \end{vmatrix} = \{EI(0)EI(l)\}^2\Delta[\varphi],$$

does not vanish by the stability condition (4.2), we get Eq. (2.1). Thus $f(x)$ is a structurally balanced loading system.

It is interesting to point out from the above argument that the functions (2.3) are also linearly independent and they exist only when the stability condition (4.2) holds.

EXAMPLE II: To find the auxiliary functions for a fixed-ended uniform beam column under the following condition:

$$EI(x) = \text{constant}, \qquad P(x) = \text{constant} > 0, \qquad k(x) = 0. \tag{4.4}$$

SOLUTION: The differential equation (1.1) becomes

$$-EIy^{iv} - Py'' = F(x).$$

Hence the auxiliary functions are

$$1, \qquad \xi, \qquad \sin \xi/j, \qquad \cos \xi/j, \tag{4.5}$$

respectively, where $j^2 = EI/P$. And the stability condition (4.2) becomes

$$\Delta[\varphi] = \frac{4}{j}\sin\frac{l}{2j}\left(\sin\frac{l}{2j} - \frac{l}{2j}\cos\frac{l}{2j}\right) \neq 0 \tag{4.6}$$

which must hold for a stable fixed-ended beam column.

Structurally equivalent loading systems

5. Structurally equivalent loading systems.

Definition. Two loading systems, $f(x)$ and $g(x)$, are said to be *structurally equivalent* with respect to a fixed-ended beam AB if and only if they give the same reaction system at both ends (in other words, the difference, $f(x) - g(x)$, forms a structurally balanced loading system with respect to the given beam).

Generalizing this definition, two loading systems are said to be structurally equivalent with respect to a *portion* AB of a beam if they are structurally equivalent with respect to the fixed-ended beam obtained by clamping the points A and B. To be sure that two loading systems, $f(x)$ and $g(x)$, are structurally equivalent, we need[3]

$$\int_0^l \varphi_i(\xi) f(\xi)\, d\xi = \int_0^l \varphi_i(\xi) g(\xi)\, d\xi, \qquad i = 1, 2, 3, 4 \tag{5.1}$$

where the φ_i are auxiliary functions. It is understood that the stability condition (4.2) is satisfied.

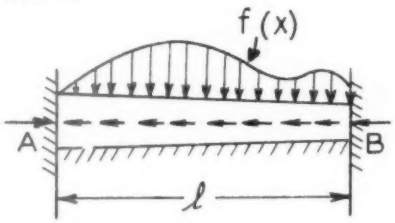

FIG. 5a.

FIG. 5b.

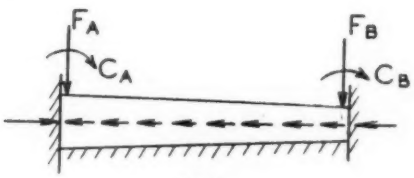

FIG. 5c.

6. Reversed reaction system. Fig. 5a shows a given loading system $f(x)$, acting on a fixed-ended beam AB. The loading system produces a reaction system as shown in Fig. 5b. Then, by definition, the *reversed reaction system* (Fig. 5c) also constitutes a loading system structurally equivalent to $f(x)$. Thus it is seen that there exists at least one loading system which is structurally equivalent to a given loading system—namely, the reversed reaction system. However, in some special cases, other useful types of structurally equivalent loading systems might be more easily obtained (see Fig. 6b).

It will now be shown how the auxiliary functions may be used to determine the

[3] The following facts are noted:

(1) If the loading $g(x)$ consists of a concentrated load F at ξ, the right-hand side of Eq. (5.1) becomes $\varphi_i(\xi)F$.

(2) If the loading $g(x)$ consists of a concentrated couple C at ξ, the right-hand side of Eq. (5.1) becomes $\varphi'_i(\xi)C$; this can easily be shown by using two concentrated loads $-C/\epsilon$ and $+C/\epsilon$ at ξ and $\xi + \epsilon$, respectively, and then letting $\epsilon \to 0$.

reversed reaction system. If $\varphi_i(\xi)$ is a set of auxiliary functions and if $\Delta[\varphi] \neq 0$, then Eq. (5.1) gives[3]

$$F_A\varphi_i(0) + F_B\varphi_i(l) + C_A\varphi_i'(0) + C_B\varphi_i'(l) = \int_0^l \varphi_i(\xi) f(\xi)\, d\xi = \eta_i\,, \qquad \text{say,} \qquad i = 1, 2, 3, 4. \tag{6.1}$$

Now if the φ_i are chosen so that

$$\left.\begin{aligned} &\varphi_2(0) = \varphi_3(0) = \varphi_4(0) = \varphi_3(l) = \varphi_4(l) = \varphi_4'(0) = 0, \\ &\varphi_1(0) = \varphi_2(l) = \varphi_3'(0) = \varphi_4'(l) = \Delta[\varphi] = 1, \end{aligned}\right\} \tag{6.2}$$

then the reversed reaction system may be readily found to be

$$C_B = \eta_4\,, \qquad C_A = \begin{vmatrix} \eta_3 & \varphi'(l) \\ \eta_4 & 1 \end{vmatrix},$$

$$\bar{F}_B = \begin{vmatrix} \eta_2 & \varphi_2'(0) & \varphi_2'(l) \\ \eta_3 & 1 & \varphi_3'(l) \\ \eta_4 & 0 & 1 \end{vmatrix}, \qquad F_A = \begin{vmatrix} \eta_1 & \varphi_1(l) & \varphi_1'(0) & \varphi_1'(l) \\ \eta_2 & 1 & \varphi_2'(0) & \varphi_2'(l) \\ \eta_3 & 0 & 1 & \varphi_3'(l) \\ \eta_4 & 0 & 0 & 1 \end{vmatrix}. \tag{6.3}$$

The above choice of φ_i is made possible by using the following linear transformation process on any given set of auxiliary functions ψ_1, ψ_2, ψ_3 and ψ_4 with $\Delta[\psi] \neq 0$:

$$\varphi_1(\xi) = \psi_1(\xi)/\psi_1(0), \qquad \varphi_2(\xi) = \begin{vmatrix} \psi_1(0) & \psi_1(\xi) \\ \psi_2(0) & \psi_2(\xi) \end{vmatrix} \div \begin{vmatrix} \psi_1(0) & \psi_1(l) \\ \psi_2(0) & \psi_2(l) \end{vmatrix}$$

$$\varphi_3(\xi) = \begin{vmatrix} \psi_1(0) & \psi_1(l) & \psi_1(\xi) \\ \psi_2(0) & \psi_2(l) & \psi_2(\xi) \\ \psi_3(0) & \psi_3(l) & \psi_3(\xi) \end{vmatrix} \div \begin{vmatrix} \psi_1(0) & \psi_1(l) & \psi_1'(0) \\ \psi_2(0) & \psi_2(l) & \psi_2'(0) \\ \psi_3(0) & \psi_3(l) & \psi_3'(0) \end{vmatrix} \tag{6.4}$$

$$\varphi_4(\xi) = \frac{1}{\Delta[\psi]} \cdot \begin{vmatrix} \psi_1(0) & \psi_1(l) & \psi_1'(0) & \psi_1'(\xi) \\ \psi_2(0) & \psi_2(l) & \psi_2'(0) & \psi_2'(\xi) \\ \psi_3(0) & \psi_3(l) & \psi_3'(0) & \psi_3'(\xi) \\ \psi_4(0) & \psi_4(l) & \psi_4'(0) & \psi_4'(\xi) \end{vmatrix}$$

Since $\Delta[\psi] \neq 0$, all denominators in Eq. (6.4) are different from zero if a proper choice of indices 1, 2, 3 and 4 is used. Thus the existence of a reversed reaction system is assured (as is physically obvious for a stable beam).

EXAMPLE III: Given a tapered beam problem with the following conditions:

$$EI(x) \neq \text{constant}, \qquad P(x) = k(x) = 0, \tag{6.5}$$

it is required (i) to find the auxiliary functions,
(ii) to find the reversed reaction system due to a unit load at ξ, and
(iii) to investigate some sufficient conditions for
a) structurally balanced loading systems, and
b) structurally equivalent loading systems
in a special case in which the reciprocal of $EI(\xi)$ is a polynomial in ξ.

SOLUTION: (i) The Eq. (1.1) becomes

$$-[EI(x)y'']'' = F(x).$$

Hence the auxiliary functions are

$$\psi_1 = 1, \qquad \psi_2 = \xi, \qquad \psi_3 = S(\xi), \qquad \psi_4 = T(\xi) \tag{6.6}$$

where

$$S(\xi) = \frac{1}{l}\int_0^\xi \int_0^\xi H(\xi)\, d\xi\, d\xi,$$

$$T(\xi) = \frac{1}{l^2}\int_0^\xi \int_0^\xi \xi H(\xi)\, d\xi\, d\xi, \tag{6.7}$$

and

$$H(\xi) = 1/EI(\xi)$$

with the value of $\Delta[\psi] = -\lambda/l \neq 0$ where λ is defined later in Eq. (6.9).

(ii) In order to find the reversed force system, we use the following auxiliary functions (according to Eq. (6.4)):

$$\varphi_1 = 1, \qquad \varphi_2 = \xi/l, \qquad \varphi_3 = \xi - \frac{lS(\xi)}{p - q},$$

$$\varphi_4 = -\frac{l}{\lambda}[(p - q)T(\xi) - (q - r)S(\xi)] \tag{6.8}$$

where the quantities p, q, r and λ defined by

$$p = \int_0^l H(\xi)\, d\xi, \qquad q = \frac{1}{l}\int_0^l \xi H(\xi)\, d\xi$$

$$r = \frac{1}{l^2}\int_0^l \xi^2 H(\xi)\, d\xi, \qquad \lambda = pr - q^2 \tag{6.9}$$

are all positive and finite. In this example the given loading system is a unit load at ξ, i.e.,

$$f(x) = \begin{cases} 0 & \text{when} \quad x \neq \xi, \\ 1 & \text{when} \quad x = \xi. \end{cases} \tag{6.10}$$

Hence we have $\eta_i = \varphi_i(\xi)$. Thus from Eq. (6.3), we obtain the reversed reaction system due to a unit load at ξ:

$$F_A = 1 - F_B\,, \qquad F_B = [qS(\xi) - pT(\xi)]/\lambda \tag{6.11}$$

and $$C_A = \xi - F_B l - C_B\,, \qquad C_B = \varphi_3\,.$$

(iiia) If $H(\xi)$, the reciprocal of $EI(\xi)$, can be expressed by a polynomial of degree n in ξ, viz.,

$$H(\xi) = \sum_{m=0}^{n} h_m(\xi/l)^m, \tag{6.12}$$

then $S(\xi)$ and $T(\xi)$ are exactly polynomials of degrees $(n + 2)$ and $(n + 3)$ respectively. Hence under the conditions (6.5) and (6.12), we have the following *sufficient* conditions that a loading system $f(x)$ be structurally balanced:

$$\int_0^l \xi^m f(\xi)\, d\xi = 0, \qquad m = 0, 1, 2, \cdots, n + 3. \tag{6.13}$$

For instance, if $f(\xi)$ can be expressed by the sum of Legendre's polynomials of order $(n + 4)$ or higher, it constitutes a balanced loading system with respect to this fixed-ended tapered beam.

(iiib) Under the same conditions (6.5) and (6.12) as mentioned above, we may also have the following *sufficient* conditions for the structural equivalence of the loading systems $f(x)$ and $g(x)$:

$$\int_0^l \xi^m f(\xi)\, d\xi = \int_0^l \xi^m g(\xi)\, d\xi, \qquad m = 0, 1, 2, \cdots, n + 3. \tag{6.14}$$

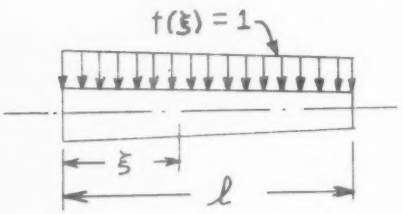

FIG. 6a.

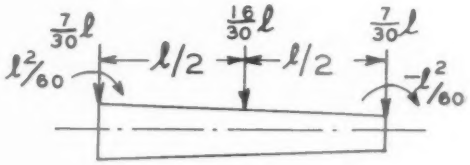

FIG. 6b.

For example, if $H(\xi)$ is a polynomial of second degree as given by

$$H(\xi) = h_0 + h_1(\xi/l) + h_2(\xi/l)^2, \tag{6.15}$$

then the two loading systems shown in Figs. 6a and 6b are structurally equivalent because they satisfy Eq. (6.14) with m = 0, 1, 2, 3, 4, and 5 (compare with the reversed reaction system shown in Fig. 6c for $h_0 = 1$, $h_1 = 0$ and $h_2 = 2$ only). A detailed discussion of various other types of structurally equivalent loading systems[4] is beyond the scope of this paper.

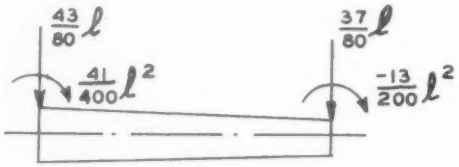

FIG. 6c.

[4]See, for example, P. P. Chen, *Dyadic analysis of space rigid frameworks*, J. Franklin Inst., Nov., 1944.

7. The equivalent loading method. In Fig. 7a is shown a loading system $f(x)$ applied to a portion AB of a beam PQ. Suppose that $g(x)$ is a loading system structurally equiv-

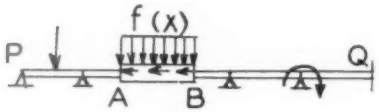

FIG. 7a.

FIG. 7b.

alent to $f(x)$ with respect to this portion AB. By definition, if the loading systems $f(x)$ and $-g(x)$ are applied simultaneously to the fixed-ended beam AB (Fig. 7b), they will produce no reaction forces and couples at the two ends A and B. Moreover, if $f(x)$ and $-g(x)$ are applied simultaneously to the portion AB of the original beam PQ (Fig. 7c), they will also produce no reaction forces and couples at any support.

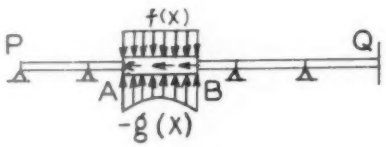

FIG. 7c.

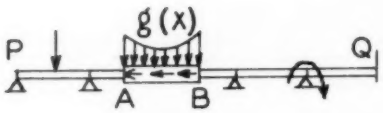

FIG. 7d.

This follows from the fact that we may consider the portion AB removed from the beam PQ, then loaded with the loading systems $f(x)$ and $-g(x)$, and then replaced in the beam PQ without having any condition of static equilibrium or geometrical continuity violated. Now consider the beam problem shown in Fig. 7d, obtained by replacing $f(x)$ in Fig. 7a by $g(x)$. Since Fig. 7a can be obtained by superposition of Figs. 7c and 7d, and since the loadings of Fig. 7c produce no reaction, it follows that the loading $g(x)$ produces the same reaction forces and couples at all supports as $f(x)$ does. Thus we have

THEOREM I: A loading system $f(x)$, when applied to a portion AB of a beam, may be replaced by a loading system $g(x)$, structurally equivalent with respect to the portion AB, without affecting the regions of the beam other than AB.

Thus the equivalent loading may be used to determine shear, moment, slope and deflection outside of AB. Within AB, these quantities can be found by superposition of the effects of the loadings in Fig. 7c and Fig. 7d.

PART II: THE EQUIVALENT BEAM METHOD
Stiffness matrix and reaction matrix

8. Fundamental simply-supported beam problem. Just as the equivalent loading method depends on the analysis of a fixed-ended beam, the equivalent beam method depends on the analysis of a simply-supported beam. The reason for this can be seen in Sec. 12.

Let us consider a horizontal simply-supported beam AB (Fig. 8) of length l loaded

by two couples C_A and C_B . It is required to find the slopes and shears at both ends. In this case the deflection, $y(x)$, is given by

$$L[y] = 0, \tag{8.1}$$

together with the boundary conditions

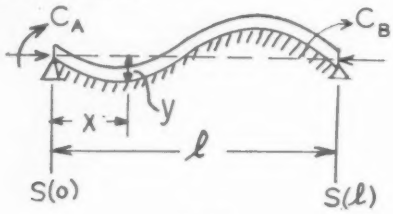

FIG. 8.

$$y(0) = y(l) = 0 \tag{8.2}$$

and

$$EI(0)y''(0) = -C_A , \qquad EI(l)y''(l) = C_B . \tag{8.3}$$

The general solution may again be expressed by

$$y(x) = \int_0^l G(x, \xi)F(\xi)\, d\xi \tag{8.4}$$

where the Green's function $G(x, \xi)$, the deflection at x due to a unit load at ξ acting on this simply-supported beam AB, is assumed to exist, and the load system $F(\xi)$ is considered as the couples C_A and C_B . Thus Eq. (8.4) gives:[3]

$$y(x) = G_\xi(x, 0)C_A + G_\xi(x, l)C_B .$$

By differentiation, we have the required end slopes:

$$\begin{aligned} y'(0) &= G_{x\xi}(0, 0)C_A + G_{x\xi}(0, l)C_B , \\ y'(l) &= G_{x\xi}(l, 0)C_A + G_{x\xi}(l, l)C_B , \end{aligned} \tag{8.5}$$

or, written in matrix form,

$$[y'(0), y'(l)] = [C_A , C_B]\begin{bmatrix} G_{x\xi}(0, 0) & G_{x\xi}(l, 0) \\ G_{x\xi}(0, l) & G_{x\xi}(l, l) \end{bmatrix}. \tag{8.6}$$

In the above equation, the square matrix

$$\begin{bmatrix} G_{x\xi}(0, 0) & G_{x\xi}(l, 0) \\ G_{x\xi}(0, l) & G_{x\xi}(l, l) \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}, \qquad \text{say,}$$

which may be called the *flexibility matrix*, is a symmetrical one due to the fact that the Green's function is symmetrical with respect to x and ξ. Its existence, as will be discussed later, depends on the fulfilment of the stability condition (9.6) for a simply-supported beam. However, its reciprocal matrix,

$$\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}^{-1} = \begin{bmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\beta} & \bar{\gamma} \end{bmatrix}, \qquad \text{say,} \tag{8.7}$$

which may be called the *stiffness matrix*,[5] is more useful in the moment distribution method. It is also a symmetrical matrix, but its existence depends on the fulfilment of stability condition (4.2) for a fixed-ended beam instead of (9.6) which is the stability condition for a simply-supported beam.

EXAMPLE IV: For a simply-supported uniform beam under the condition (2.4), the Green's function is

$$G(x, \xi) = \begin{cases} x(l - \xi)(2l\xi - \xi^2 - x^2)/6EIl, & x < \xi, \\ \xi(l - x)(2lx - x^2 - \xi^2)/6EIl, & \xi < x. \end{cases} \tag{8.8}$$

Here the flexibility and stiffness matrices are, respectively,

$$\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \frac{l}{6EI}, \qquad \begin{bmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\beta} & \bar{\gamma} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \frac{2EI}{l}. \tag{8.9}$$

The end shears may be obtained from the static condition

$$S(0) = S(l) = -(C_A + C_B)/l. \tag{8.10}$$

9. Stiffness and reaction matrices in terms of auxiliary functions. The general solution of Eq. (8.1) may also be expressed in terms of auxiliary functions $\varphi_i(x)$:

$$y(x) = \sum_{i=1}^{4} a_i \varphi_i(x). \tag{9.1}$$

From Eq. (9.1) and the boundary conditions (8.2) and (8.3), the coefficients a_i may be determined. Again if the φ_i functions given by Eq. (6.4) are used, we have,

$$\begin{aligned} &a_1 = a_2 = 0, \\ &y'(0) = a_3 , \\ &y'(l) = a_3\varphi_3'(l) + a_4 , \\ &y''(0) = -C_A/EI(0) = a_3\varphi_3''(0) + a_4\varphi_4''(0), \\ \text{and} \quad &y''(l) = C_B/EI(0) = a_3\varphi_3''(l) + a_4\varphi_4''(l), \end{aligned} \tag{9.2}$$

provided that the stability condition (4.2) holds. Eliminating a_3 and a_4, we have

$$[C_A , C_B] = [y'(0), y'(l)] \begin{bmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\beta} & \bar{\gamma} \end{bmatrix} \tag{9.3}$$

where the elements in the stiffness matrix are given by

$$\begin{aligned} \bar{\alpha} &= [-\varphi_3''(0) + \varphi_3'(l)\varphi_4''(0)]EI(0), \\ \bar{\beta} &= -\varphi_4''(0)EI(0) = [\varphi_3''(l) - \varphi_3'(l)\varphi_4''(l)]EI(l), \\ \text{and} \quad \bar{\gamma} &= \varphi_4''(l)EI(l). \end{aligned} \tag{9.4}$$

[5] S. U. Benscoter, *Matrix analysis of continuous beams*, Trans. A. S. C. E. 112, 1109 (1947).

The non-vanishing of the determinant of the stiffness matrix,

$$\bar{\alpha}\bar{\gamma} - \bar{\beta}^2 = - \begin{vmatrix} \varphi_3''(0) & \varphi_3''(l) \\ \varphi_4''(0) & \varphi_4''(l) \end{vmatrix} EI(0)EI(l) \neq 0, \tag{9.5}$$

where the φ_i are given by Eq. (6.4), or, more generally, the non-vanishing of the following determinant,

$$D[\varphi] = \begin{vmatrix} \varphi_1(0) & \varphi_1(l) & \varphi_1''(0) & \varphi_1''(l) \\ \varphi_2(0) & \varphi_2(l) & \varphi_2''(0) & \varphi_2''(l) \\ \varphi_3(0) & \varphi_3(l) & \varphi_3''(0) & \varphi_3''(l) \\ \varphi_4(0) & \varphi_4(l) & \varphi_4''(0) & \varphi_4''(l) \end{vmatrix} \neq 0, \tag{9.6}$$

where the φ_i are linearly independent auxiliary functions, gives the stability condition for a simply-supported beam problem.[6]

To find the end shears, we start from Eq. (1.3),

$$\begin{aligned} S(x) &= -\{EI(x)y''(x)\}' - P(x)y'(x) \\ &= -\sum_{i=3,4} a_i\{EI(x)\varphi_i''(x)\}' - P(x)y'(x) \end{aligned} \tag{9.7}$$

where the coefficients a_i may be obtained from Eq. (9.2):

$$[a_3\ ,\ a_4] = [y'(0),\ y'(l)] \begin{bmatrix} 1 & -\varphi_3'(l) \\ 0 & 1 \end{bmatrix}. \tag{9.8}$$

Putting $x = 0$ and l respectively in Eq. (9.7), we have the end shears

$$[S(0),\ S(l)] = [y'(0),\ y'(l)] \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix} \tag{9.9}$$

where

$$\begin{aligned} \epsilon_{11} &= -P(0) - \{EI\varphi_3''\}'(0) + \varphi_3'(l)\{EI\varphi_4''\}'(0), \\ \epsilon_{12} &= -\{EI\varphi_3''\}'(l) + \varphi_3'(l)\{EI\varphi_4''\}'(l), \\ \epsilon_{21} &= -\{EI\varphi_4''\}'(0), \\ \epsilon_{22} &= -P(l) - \{EI\varphi_4''\}'(l). \end{aligned}$$

If the relation (9.6) holds, then the end shears can also be expressed by

$$[S(0),\ S(l)] = [C_A\ ,\ C_B] \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \tag{9.10}$$

[6]For example, for a stable simply-supported beam column under the condition (4.4), the stability condition (9.6) gives $D[\varphi] = -(l/j^4)\sin(l/j) \neq 0$. It is interesting to point out that if a portion AB of a beam is such that $D[\varphi] = 0$, then this portion acts as a hinged link.

where the square matrix

$$\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix} \tag{9.11}$$

may be called the *reaction matrix*.

However if the beam is under the conditions

$$k(x) = 0, \qquad P(x) = \text{constant}, \tag{9.12}$$

then we also obtain Eq. (8.10) even if the relation (9.6) does not hold. Thus the reaction matrix has the following simple form:

$$\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = -\frac{1}{l} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{9.13}$$

with the distinctive properties that

(i) it depends on the length l only

and (ii) all elements ρ_{11} , ρ_{12} , ρ_{21} and ρ_{22} are equal.

10. Simply supported beams with two or more segments under the conditions $k(x) = 0$ and $P(x) =$ constant. In Fig. 9 is shown a simply-supported beam of two

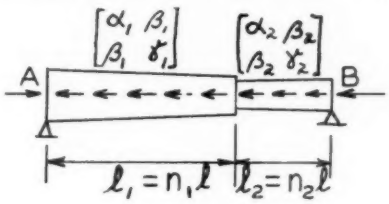

FIG. 9.

segments AG and GB, of lengths l_1 and l_2 respectively. If the flexibility matrices are $\begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_1 & \gamma_1 \end{bmatrix}$ and $\begin{bmatrix} \alpha_2 & \beta_2 \\ \beta_2 & \gamma_2 \end{bmatrix}$, respectively, it can be shown that the elements α, β and γ of the resultant flexibility matrix $\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$ of the whole beam are:

$$\begin{aligned} \alpha &= \alpha_1 - 2n_2\beta_1 + n_2^2(\gamma_1 + \alpha_2), \\ \beta &= n_1\beta_1 + n_2\beta_2 - n_1n_2(\gamma_1 + \alpha_2), \\ \text{and} \qquad \gamma &= \gamma_2 - 2n_1\beta_2 + n_1^2(\gamma_1 + \alpha_2). \end{aligned} \tag{10.1}$$

By repeated application of Eq. (10.1), the flexibility matrix of a beam of any finite number of segments under the conditions (9.12) can be calculated.

Various types of equivalent beams

11. Equivalent beams under the conditions (9.12).

Definition. Two horizontal simply-supported beams AB and $A'B'$ (Figs. 10a and 10b) are said to be *equivalent*, if and only if they have the same stiffness and reaction matrices.

Generalizing this definition, two beam *portions* are said to be equivalent if they are equivalent when considered as simply-supported beams.

If these beam portions are not elastically supported, and if $P(x)$ = constant, then they need only be of equal length in order to have the same reaction matrix (see Eq. 9.13). In this paragraph we consider only this situation, unless otherwise specified.

Let us investigate the following types of equivalent beams: (Type I) Let the beam AB (Fig. 10a) have a given flexibility matrix $\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$, and let us find an equivalent beam $A'B'$ (Fig. 10b) composed of two uniform segments $A'G'$ and $G'B'$ with lengths

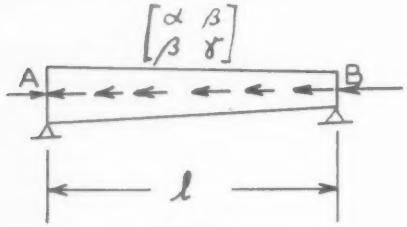

FIG. 10a.

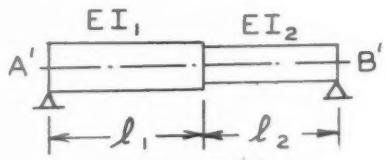

FIG. 10b.

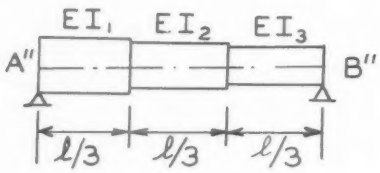

FIG. 10c.

l_1 and $l_2(=l - l_1)$ and rigidities EI_1 and EI_2 respectively. Equating the flexibility matrix of beam $A'B'$ (as given by Eq. (10.1)) to that of beam AB, we get

$$\left.\begin{aligned} l_1 &= \frac{\gamma + 2\beta}{\gamma - \alpha}\, l, \qquad l_2 = \frac{\alpha + 2\beta}{\alpha - \gamma}\, l \\ \text{and} \qquad EI_1 &= \frac{l}{3\alpha + (\alpha + 2\beta) l_2/l_1}, \qquad EI_2 = \frac{l}{3\gamma + (\gamma + 2\beta) l_1/l_2} \end{aligned}\right\} \tag{11.1}$$

provided that $0 \leqq l_1 \leqq l$,
that is,

$$\alpha \gtreqless -2\beta \gtreqless \gamma. \tag{11.2}$$

(Type II) Assume that the beam AB has a given flexibility matrix $\begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$, and find an equivalent beam $A''B''$ (Fig. 10c) composed of three parts of equal length but different rigidities EI_1, EI_2 and EI_3. We get

$$\left.\begin{aligned} EI_1 &= 2l/(11\alpha + 2\gamma + 14\beta), \\ EI_2 &= -2l/(7\alpha + 7\gamma + 40\beta), \\ \text{and} \qquad EI_3 &= 2l/(11\gamma + 2\alpha + 14\beta). \end{aligned}\right\} \tag{11.3}$$

Hence under the conditions $k(x) = 0$ and $P(x) =$ constant, it is always possible to find an equivalent beam composed of only three uniform segments, if the use of zero or negative values of EI is permitted.

However, if (9.12) is not satisfied, the elements ρ_{11} , ρ_{12} , ρ_{21} and ρ_{22} in the reaction matrix are, in general, not equal to each other. Hence, we cannot find an equivalent portion $A'B'$ for the original portion AB (Fig. 11a) unless some additional supports are provided (see Fig. 11b). A detailed discussion of equivalent beams for this case is not given here.

EXAMPLE V: To find the equivalent beams for a tapered beam (Fig. 12a) under the condition (6.5).

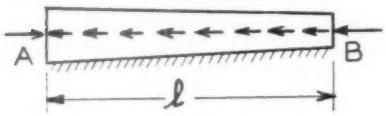

FIG. 11a.

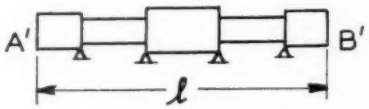

FIG. 11b.

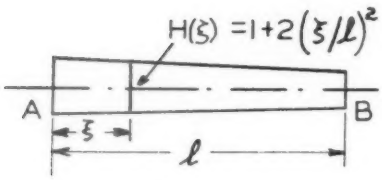

FIG. 12a.

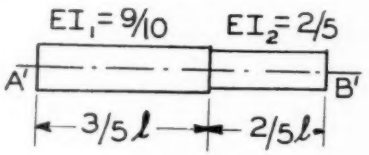

FIG. 12b.

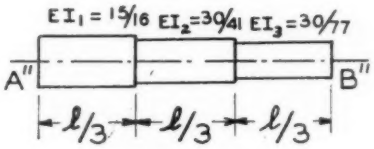

FIG. 12c.

SOLUTION: Use the auxiliary functions (6.8).
From Eqs. (9.4), (8.7) and (9.12), we have

$$\bar{\alpha} = r/\lambda, \qquad \bar{\beta} = (q - r)/\lambda, \qquad \bar{\gamma} = (p - 2q + r)/\lambda$$

$$\alpha = p - 2q + r, \qquad \beta = r - q, \qquad \gamma = r \tag{11.4}$$

and $$\rho_{11} = \rho_{12} = \rho_{21} = \rho_{22} = -1/l.$$

As a numerical example, we assume that $H(\xi)$ of the beam (Fig. 12a) is given by

$$H(\xi) = 1 + 2(\xi/l)^2. \tag{11.5}$$

From Eqs. (6.9) and (11.4), we have

$$p = 5l/3, \qquad q = l, \qquad r = \gamma = 11l/15, \qquad \alpha = 2l/5, \qquad \beta = -4l/15.$$

Then by Eqs. (11.1) and (11.3), the equivalent beams may be obtained as shown in Figs. 12b and 12c.

12. The equivalent beam method and an illustrative example. In Fig. 13a is shown a loaded beam with a portion AB which is free from lateral load. In Fig. 13b is shown the same loaded beam except that the portion AB is replaced by its equivalent one, $A'B'$. By reasoning similar to that used in Sec. 7, we have

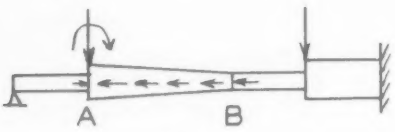

FIG. 13a.

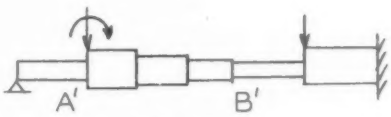

FIG. 13b.

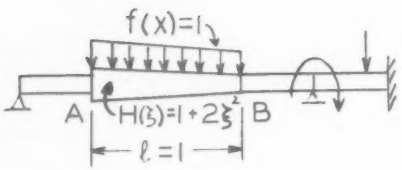

FIG. 14a.

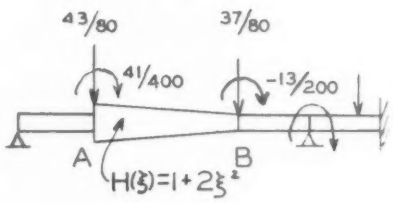

FIG. 14b.

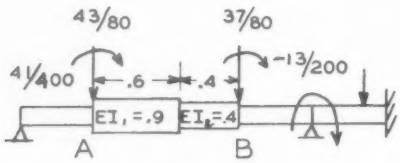

FIG. 14c.

THEOREM II. In any beam problem, a portion AB, if free from lateral load, may be replaced by an equivalent portion without affecting the regions of the beam other than AB.

As a final illustration of the method presented, we will show the application to the beam shown in Fig. 14a.

EXAMPLE VI. To simplify the beam problem shown in Fig. 14a by using both equivalent loading and equivalent beam methods (use $l = 1$ for convenience).

SOLUTION: First, using the equivalent loading method to remove the distributed loads (see Figs. 6a and 6c), we get Fig. 14b.

Secondly, applying the equivalent beam method to the portion AB in Fig. 14b to remove the varying cross sections (see Figs. 12a and 12b), we get Fig. 14c. Then the analysis of the beam in Fig. 14c will give precise results for the beam in Fig. 14a everywhere other than the portion AB. Thus we may at least obtain the precise results at the ends A and B of the affected portion. These precise results might be then used in the determination of exact local value of shear, moment, slope and deflection within the portion AB.

By using these methods, we obtain the linearized M/EI diagrams which greatly facilitate the subsequent computations.

A NEW THEORY OF PLASTIC FLOW*

BY

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1. Introduction. This paper is concerned with a new mathematical theory of plastic flow for an isotropic, incompressible strain-hardening material exhibiting a gradual transition from the elastic to the plastic state. The discussion is restricted to states of stress and strain which can be reached by a single loading, followed by at most one partial or complete unloading.

The new stress-strain relations are introduced in Sec. 2, and the criteria for loading and unloading are established. The boundary value problem for the strain rates corresponding to given surface velocities is formulated in Sec. 3 and uniqueness of solution is proved. Sections 4 and 5 are concerned with the minimum principle associated with this boundary value problem. Sections 6 and 7 bring a comparison between the new theory and the more familiar theory of plastic deformation.[1] Finally, the conditions under which the two theories furnish identical results are investigated in Sec. 8.

2. Stress-strain relation. The customary form of the stress-strain relation of the flow theory for strain-hardening materials [2, Eq. 27] gives the rate of change of the strain tensor as a function of the stress tensor, the rate of change of the stress tensor, and the stress invariants. The treatment of some problems of practical interest is greatly facilitated, however, by the use of stress-strain relations giving the rate of change of the stress tensor explicitly in terms of the strain tensor, the rate of change of the strain tensor, and the strain invariants. Equations of this form cannot be obtained, as a rule, from the customary stress-strain relations of the flow theory. We thus are led to consider the following stress-strain relation for an incompressible material:[2]

$$s_{ij}^* = 2G_0\epsilon_{ij}^* - p(E)E^*\epsilon_{ij} \ . \tag{2.1}$$

Here the stress deviation s_{ij} is defined as follows in terms of the stress tensor σ_{ij} :

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\,\delta_{ij}\sigma_{kk} \ .$$

As usual, repeated indices indicate summation in accordance with the summation convention of tensor calculus; δ_{ij} is the Kronecker delta, ϵ_{ij} is the strain tensor, $E = 2\epsilon_{ij}\epsilon_{ij}/3$ the strain intensity, G_0 the shear modulus in the elastic range, and $p = p(E)$ an arbitrary function depending on the material. Variables with asterisks indicate the rates of change of these variables with respect to a parameter t which may be thought of as time.[3] Stress-strain relations of the type (2.1) may be called *stress theories of plastic*

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[1]This terminology was introduced by A. A. Ilyushin [1]. (Numbers in square brackets refer to the bibliography at the end of the paper).

[2]The author is indebted to W. Prager for the form of this relation which was introduced in [3].

[3]Since Eq. (2.1) is homogeneous in the rates of change, no viscosity effects are introduced, and the equation remains unchanged if t is replaced by any monotonically increasing function t, i.e. by any other parameter which increases with increasing t.

flow in contrast to *strain theories of plastic flow* such as are treated in [2, Sec. 2]. Equation (2.1) is to apply for "loading" only. The precise meaning of this term for states of combined stress will be given later.

The arbitrary function $p(E)$ in Eq. (2.1) can be determined for a particular material from the stress-strain diagram in simple tension. Denoting the tensile stress by σ and the corresponding unit extension by ϵ, the results of such a test for the type of material being considered here is given in Fig. 1. The stress-strain relation (2.1) reduces to the single equation

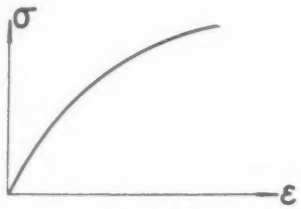

FIG. 1

$$p(E) = \left(G_0 - \frac{1}{3}\frac{d\sigma}{d\epsilon}\right)/E, \tag{2.2}$$

while the strain intensity becomes $E = \epsilon^2$. Assuming that $\sigma(\epsilon)$ can be represented with sufficient accuracy by the power series

$$\sigma = a_1\epsilon + a_3\epsilon^3 + a_5\epsilon^5 + \cdots, \tag{2.3}$$

where $a_1 = 3G_0$, and a_3, a_5, $\cdots$ are constants, it follows from Eqs. (2.2) and (2.3) that $p(E)$ is given by

$$p(E) = b_1 + b_2E + b_3E^2 + \cdots, \tag{2.4}$$

where $b_n = -(2n + 1)\, a_{2n+1}/3$, $n = 1, 2, 3, \cdots$. The stress-strain relation (2.1) for loading then becomes

$$s^*_{ij} = 2G_0\epsilon^*_{ij} - [b_1 + b_2E + b_3E^2 + \cdots]E^*\epsilon_{ij}. \tag{2.5}$$

The stress-strain relation for unloading will be assumed to be given by the differentiated form of Hooke's law:

$$s^*_{ij} = 2G_0\epsilon^*_{ij}. \tag{2.6}$$

Under combined states of stress and strain, loading may occur in certain portions of a body and unloading in others, so that it becomes necessary to establish some sort of loading-unloading criterion. For that purpose we will assume that loading takes place wherever $\epsilon_{ij}d\sigma_{ij} > 0$, and unloading wherever $\epsilon_{ij}d\sigma_{ij} < 0$. If we multiply Eq. (2.1) by the strain tensor, we find that

$$\sigma^*_{ij}\epsilon_{ij} = \frac{3}{2}E^*(G_0 - Ep(E)) > 0 \tag{2.7}$$

for loading. On the other hand, all along the stress-strain curve of Fig. 1 the slope satisfies

$$0 < \frac{d\sigma}{d\epsilon} \leq 3G_0,$$

and therefore, by Eq. (2.2),

$$0 \leq p(E) < \frac{G_0}{E}. \tag{2.8}$$

It follows from Eq. (2.7) and (2.8) that $E^* > 0$ for loading. A similar procedure with Eq. (2.6) shows that the unloading criterion is $E^* < 0$.

The stress-strain relations for loading and unloading can be combined into one expression, namely

$$s_{ij}^* = 2G_0\epsilon_{ij}^* - \tfrac{1}{2}p(E)\epsilon_{ij}(E^* + \mid E^* \mid) \tag{2.9}$$

which reduces to Eq. (2.1) when $E^* > 0$ and to (2.6) when $E^* < 0$.

3. Uniqueness of solution. With the aid of two lemmas which are to be established in this section the following uniqueness theorem will be proved.

Theorem. For a given state of strain ϵ_{ij} throughout the body, and given velocities u_i^* on the surface such that the surface integral of the normal velocity component vanishes, the rate of change of strain ϵ_{ij}^* is *uniquely* determined throughout the body.

Lemma 1. Let ϵ_{ij} be some fixed state of strain throughout the body, and $\epsilon_{ij}^{*\prime}$ and $\epsilon_{ij}^{*\prime\prime}$ two arbitrary sets of strain rates which satisfy the condition of incompressibility. Furthermore, let $\sigma_{ij}^{*\prime}$ and $\sigma_{ij}^{*\prime\prime}$ be the corresponding stress rates obtained from the stress-strain relation (2.9). Then

$$(\epsilon_{ij}^{*\prime} - \epsilon_{ij}^{*\prime\prime})(\sigma_{ij}^{*\prime} - \sigma_{ij}^{*\prime\prime}) > 0$$

holds throughout the body independent of whether there is local loading or unloading.

Proof. Setting $d_{ij}^* = \epsilon_{ij}^{*\prime} - \epsilon_{ij}^{*\prime\prime}$ and noting that $d_{ij}^*(\sigma_{ij}^{*\prime} - \sigma_{ij}^{*\prime\prime}) = d_{ij}^*(s_{ij}^{*\prime} - s_{ij}^{*\prime\prime})$, we find from Eq. (2.9), that

$$d_{ij}^*(s_{ij}^{*\prime} - s_{ij}^{*\prime\prime}) = 2G_0 d_{ij}^* d_{ij}^* - \tfrac{1}{2}p(E)d_{ij}^*\epsilon_{ij}\{E^{*\prime} - E^{*\prime\prime} + \mid E^{*\prime} \mid - \mid E^{*\prime\prime} \mid\}.$$

Because

$$\mid \{E^{*\prime} - E^{*\prime\prime} + \mid E^{*\prime} \mid - \mid E^{*\prime\prime} \mid\} \mid \leq \mid E^{*\prime} - E^{*\prime\prime} \mid + \mid \mid E^{*\prime} \mid - \mid E^{*\prime\prime} \mid \mid$$

$$\leq 2 \mid E^{*\prime} - E^{*\prime\prime} \mid$$

and

$$E^{*\prime} - E^{*\prime\prime} = \frac{4}{3}\epsilon_{rs}d_{rs}^*,$$

we have

$$d_{ij}^*(s_{ij}^{*\prime} - s_{ij}^{*\prime\prime}) \geq 2G_0 d_{ij}^* d_{ij}^* - \frac{4}{3}p(E)(d_{ij}^*\epsilon_{ij})^2$$

since $p(E) \geq 0$ by Eq. (2.8). By the Schwarzian inequality

$$(d_{ij}^*\epsilon_{ij})^2 \leq d_{ij}^* d_{ij}^* \epsilon_{rs}\epsilon_{rs} = \frac{3}{2}d_{ij}^* d_{ij}^* E$$

so that

$$d_{ij}^*(s_{ij}^{*\prime} - s_{ij}^{*\prime\prime}) \geq 2d_{ij}^* d_{ij}^*\{G_0 - Ep(E)\} > 0.$$

This proves the lemma (see Eq. (2.8)).

Lemma 2. Let V be any "regular region" [4, p. 112] with surface S which is the sum of a finite number of "normal regions" [4, p. 85], V_1 , V_2 , $\cdots$, V_n , and $\sigma_{ij}^{*\prime}$ and $\epsilon_{ij}^{*\prime\prime}$ functions satisfying the following conditions:

1) in each normal region
   a) $\sigma_{ij}^{*\prime} = \sigma_{ji}^{*\prime}$,
   b) $\sigma_{ij}^{*\prime}$ is continuous and differentiable,
   c) $\sigma_{ij,k}^{*\prime} \equiv \partial\sigma_{ij}^{*\prime}/\partial x_k$ is continuous,
   d) $\sigma_{ij,i}^{*\prime} = 0$ (equation of equilibrium);

2) the rates of change of the stresses transmitted across the surface of separation between two adjacent normal regions are the same whether evaluated from $\sigma_{ij}^{*\prime}$ on one side or $\sigma_{ij}^{*\prime}$ on the other;

3) in each normal region
   a) $\epsilon_{ij}^{*\prime\prime}$ is continuous,
   b) $\epsilon_{ij}^{*\prime\prime} = 1/2(u_{i,j}^{*\prime\prime} + u_{j,i}^{*\prime\prime})$; and

4) the velocities $u_i^{*\prime\prime}$ are continuous throughout $V + S$.

Then

$$\int_V \sigma_{ij}^{*\prime}\epsilon_{ij}^{*\prime\prime}\, dv = \int_S \sigma_{ij}^{*\prime}u_i^{*\prime\prime}n_j\, dS,$$

where n_j is the external normal to the surface.

Proof. Using the conditions (1b), (3a), (3b), (1a), (1d), we find

$$\int_V \sigma_{ij}^{*\prime}\epsilon_{ij}^{*\prime\prime}\, dv = \sum_1^n \int_{V_k} \sigma_{ij}^{*\prime}\epsilon_{ij}^{*\prime\prime}\, dv = \sum_1^n \int_{V_k} (\sigma_{ij}^{*\prime}u_i^{*\prime\prime})_{,j}\, dv.$$

Use of conditions (1c), (2), (4), and the divergence theorem then gives us

$$\sum_1^n \int_{V_k} (\sigma_{ij}^{*\prime}u_i^{*\prime\prime})_{,j}\, dv = \int_S \sigma_{ij}^{*\prime}u_i^{*\prime\prime}n_j\, dS.$$

It is now possible to prove the uniqueness theorem stated earlier. Let us assume that two solutions ϵ_{ij}^{*} and $\epsilon_{ij}^{*\prime}$ exist each deriving from a system of velocities in accordance with

$$\epsilon_{ij}^{*} = \tfrac{1}{2}(u_{i,j}^{*} + u_{j,i}^{*})$$

and

$$\epsilon_{ij}^{*\prime} = \tfrac{1}{2}(u_{i,j}^{*\prime} + u_{j,i}^{*\prime}),$$

where $u_i^{*} = u_i^{*\prime}$ on the boundary. Since σ_{ij}^{*} and $\sigma_{ij}^{*\prime}$, the corresponding rates of change of stress obtained from Eq. (2.9), represent solutions, they must satisfy the equations of equilibrium, $\sigma_{ij,i}^{*} = 0$ and $\sigma_{ij,i}^{*\prime} = 0$. It is assumed that no body forces exist. All the conditions of Lemma 2 being satisfied, we can conclude that

$$\int_V (\epsilon_{ij}^{*} - \epsilon_{ij}^{*\prime})(\sigma_{ij}^{*} - \sigma_{ij}^{*\prime})\, dv = \int_S (u_i^{*} - u_i^{*\prime})(\sigma_{ij}^{*} - \sigma_{ij}^{*\prime})n_j\, dS.$$

But since on the surface $u_i^{*} = u_i^{*\prime}$ the surface integral vanishes and hence the volume integral also. On the other hand, by Lemma 1 the volume integral is greater than zero, unless $\epsilon_{ij}^{*} = \epsilon_{ij}^{*\prime}$. Thus, the solution ϵ_{ij}^{*} must be unique.

4. A minimum principle for loading throughout the body. We shall confine ourselves in this section to the case in which loading takes place throughout the entire body. We denote existing strains by ϵ_{ij} and the solution, i.e., the strain rates which actually occur, by ϵ_{ij}^*. These strain rates must satisfy the following:

1) the equation of equilibrium $\sigma_{ij,j}^* = 0$,

2) $\epsilon_{kk}^* = 0$,

3) $\epsilon_{ij}^* = \frac{1}{2}(u_{i,j}^* + u_{j,i}^*)$, and

4) $\epsilon_{ij}\epsilon_{ij}^* > 0$,

where u_i^* equals the prescribed boundary values on the surface S. Let $\epsilon_{ij}^{*\prime}$ be an arbitrary set of strain rates which satisfy the conditions:

1) $\epsilon_{kk}^{*\prime} = 0$,

2) $\epsilon_{ij}^{*\prime} = \frac{1}{2}(u_{i,j}^{*\prime} + u_{j,i}^{*\prime})$, and

3) $\epsilon_{ij}\epsilon_{ij}^{*\prime} > 0$,

where $u_i^{*\prime} = u_i^*$ on the surface. With these restrictions on the natural strain rates ϵ_{ij}^* and the admissible strain rates $\epsilon_{ij}^{*\prime}$ the following minimum principle can be established.

Minimum Principle.

$$\int_V \epsilon_{ij}^*\sigma_{ij}^*\,dv \leq \int_V \epsilon_{ij}^{*\prime}\sigma_{ij}^{*\prime}\,dv$$

when σ_{ij}^* and $\sigma_{ij}^{*\prime}$ are obtained from ϵ_{ij}^* and $\epsilon_{ij}^{*\prime}$, respectively, by means of the stress-strain relation (2.9). Equality holds if and only if $\epsilon_{ij}^* = \epsilon_{ij}^{*\prime}$.

In proving this principle we use the following lemma.

Lemma 3. For the stress rates and strain rates introduced above

$$\epsilon_{ij}^*\sigma_{ij}^{*\prime} = \epsilon_{ij}^{*\prime}\sigma_{ij}^* .$$

Proof. From $\epsilon_{ij}^*\sigma_{ij}^{*\prime} \equiv \epsilon_{ij}^* s_{ij}^{*\prime}$ and $\epsilon_{ij}^{*\prime}\sigma_{ij}^* \equiv \epsilon_{ij}^{*\prime} s_{ij}^*$, the lemma is obtained by applying Eq. (2.9) for $E^* > 0$ and $E^{*\prime} > 0$. The equality also holds when $E^* < 0$ and $E^{*\prime} < 0$. We note that Lemma 3 would not hold if, for instance, ϵ_{ij}^* were to constitute loading and $\epsilon_{ij}^{*\prime}$ unloading.

Returning to the minimum principle it follows from Lemma 1 that

$$\int_V \epsilon_{ij}^*\sigma_{ij}^*\,dv - \int_V \epsilon_{ij}^{*\prime}\sigma_{ij}^*\,dv - \int_V \epsilon_{ij}^*\sigma_{ij}^{*\prime}\,dv + \int_V \epsilon_{ij}^{*\prime}\sigma_{ij}^{*\prime}\,dv \geq 0, \tag{4.1}$$

where equality holds if and only if $\epsilon_{ij}^* = \epsilon_{ij}^{*\prime}$. According to Lemma 2,

$$\int_V \epsilon_{ij}^{*\prime}\sigma_{ij}^*\,dv = \int_S u_i^{*\prime}\sigma_{ij}^* n_j\,dS;$$

and

$$\int_V \epsilon_{ij}^*\sigma_{ij}^*\,dv = \int_S u_i^*\sigma_{ij}^* n_j\,dS.$$

Since $u_i^{*\prime} = u_i^*$ on S it follows that

$$\int_V \epsilon_{ij}^{*\prime}\sigma_{ij}^* \, dv = \int_V \epsilon_{ij}^*\sigma_{ij}^* \, dv. \tag{4.2}$$

By Lemma 3 and (4.2), we thus have

$$\int_V \epsilon_{ij}^*\sigma_{ij}^{*\prime} \, dv = \int_V \epsilon_{ij}^*\sigma_{ij}^* \, dv. \tag{4.3}$$

Substitution of relations (4.2) and (4.3) into (4.1) completes the proof.

5. Extension of the minimum principle. For the case where loading occurs in a sub-domain of the body and unloading occurs everywhere else,[4] some of the arguments presented in Sec. 4 are no longer valid because Lemma 3 cannot be applied throughout the body. Since the actual surface of separation between the two domains of loading and unloading for ϵ_{ij}^* is not known, the surface which separates the regions of loading and unloading for $\epsilon_{ij}^{*\prime}$ is in general distinct from the actual surface. Accordingly, the domain V is divided by these surfaces into what we assume to be four normal sub-domains, namely V_{++}, V_{-+}, V_{+-}, and V_{--}; the first of the two subscripts refers to the actual state, and the second to the artificial state, the plus sign denoting loading and the negative sign unloading.

Both (4.1) and (4.2) are still valid in this more general case, so that

$$\int_V \epsilon_{ij}^*\sigma_{ij}^{*\prime} \, dv \leq \int_V \epsilon_{ij}^{*\prime}\sigma_{ij}^{*\prime} \, dv. \tag{5.1}$$

Since Lemma 3 applies over the domains V_{++} and V_{--}, we have

$$\int_V \epsilon_{ij}^*\sigma_{ij}^{*\prime} \, dv = \int_{V_{++}} \epsilon_{ij}^{*\prime}\sigma_{ij}^* \, dv + \int_{V_{--}} \epsilon_{ij}^{*\prime}\sigma_{ij}^* \, dv + \int_{V_{-+}} \epsilon_{ij}^*\sigma_{ij}^{*\prime} \, dv + \int_{V_{+-}} \epsilon_{ij}^*\sigma_{ij}^{*\prime} \, dv,$$

or

$$\int_V \epsilon_{ij}^*\sigma_{ij}^{*\prime} \, dv = \int_V \epsilon_{ij}^{*\prime}\sigma_{ij}^* \, dv + \int_{V_{-+}} L \, dv + \int_{V_{+-}} L \, dv, \tag{5.2}$$

where

$$L = \epsilon_{ij}^*\sigma_{ij}^{*\prime} - \epsilon_{ij}^{*\prime}\sigma_{ij}^* \equiv \epsilon_{ij}^* s_{ij}^{*\prime} - \epsilon_{ij}^{*\prime} s_{ij}^* \, .$$

By use of Eq. (2.9), we find that

$$L = \frac{3}{8} p(E)\{E^{*\prime} \mid E^* \mid - E^* \mid E^{*\prime} \mid\}. \tag{5.3}$$

Using Eqs. (5.2), (5.3), (4.2) and the fact that $E^* < 0$, $E^{*\prime} > 0$ in V_{-+} and $E^* > 0$, $E^{*\prime} < 0$ in V_{+-}, we write (5.1) in the form

$$\int_V \epsilon_{ij}^*\sigma_{ij}^* \, dv - \frac{3}{4}\int_{V_{-+}} p(E)E^*E^{*\prime} \, dv + \frac{3}{4}\int_{V_{+-}} p(E)E^*E^{*\prime} \, dv \leq \int_V \epsilon_{ij}^{*\prime}\sigma_{ij}^{*\prime} \, dv.$$

In the domains V_{-+} and V_{+-}, we have $pE^*E^{*\prime} \leq 0$ since $p(E) \geq 0$ (see (2.8)).

[4] It is assumed here that the domains of loading and unloading separate the body into two normal regions. This condition is not necessary and is introduced for the purpose of simplification only. The generalization to any finite number of normal regions is immediate.

We conclude that the inequality

$$\int_V \epsilon_{ij}^* \sigma_{ij}^* \, dv \leq \int_V \epsilon_{ij}^{*\prime} \sigma_{ij}^{*\prime} \, dv$$

holds even in this more general case, whenever the artificial state is made to constitute loading in those regions where the actual state constitutes loading. In this case there is no V_{+-} domain.[5]

6. Stress-strain relations for a theory of plastic deformation. The customary stress-strain relations [6, p. 75] for incompressible materials possessing the property of gradual transition from the elastic to the plastic state are

$$s_{ij} = 2G' \epsilon_{ij} , \tag{6.1}$$

where $G' = G'(E)$ is the *secant shear modulus*. The secant shear modulus, being a function of the strain intensity E alone, can be evaluated for any material in the manner in which the quantity $p(E)$ was obtained for the flow theory.

For simple tension Eq. (6.1) reduces to the single expression

$$\sigma/\epsilon = 3G'.$$

Using the same series expansion (2.3) as in the plastic flow theory, we obtain

$$3G' = 3G_0 + a_3 E + a_5 E^2 + \cdots . \tag{6.2}$$

Substituting this into Eq. (6.1) we find

$$s_{ij} = 2G_0 \epsilon_{ij} - [c_1 + c_2 E + c_3 E^2 + \cdots] E \epsilon_{ij} , \tag{6.3}$$

where $c_n = -2a_{2n+1}/3$ and $n = 1, 2, 3, \cdots$.

7. Comparison between the two plastic theories. Consider a solid composed of a strain-hardening material which is deformed from a state of zero stress and strain by an equilibrium system of surface stresses such that loading occurs throughout the body. We assume that the relation between stress and strain in simple tension for the material, as given by the series expansion

$$\sigma = a_1 \epsilon + a_3 \epsilon^3 + a_5 \epsilon^5 + \cdots , \tag{2.3}$$

is known. The complete systems of equations for the two theories are then as follows.

I. *Plastic Flow Theory.*

a) Equations of equilibrium

$$\sigma_{ij,i}^* = 0,$$

where body forces are neglected;

b) condition of incompressibility

$$\epsilon_{kk}^* = 0;$$

c) stress-strain relation

$$s_{ij}^* = 2G_0 \epsilon_{ij}^* - [b_1 + b_2 E + b_3 E^2 + \cdots] E^* \epsilon_{ij} ,$$

where $b_n = -(2n + 1)a_{2n+1}/3$;

[5] For a discussion of minimum principles based on strain theory of plastic flow see [5].

d) strain rate-velocity relation

$$\epsilon_{ij}^{*} = \tfrac{1}{2}(u_{i,j}^{*} + u_{j,i}^{*}),$$

where the velocity vector u_i^* is assumed small; and

e) boundary conditions

$$u_i^{*} = U_i^{*},$$

where U_i^* is the prescribed velocity of the surface.

Equations I(a)-I(d) must be satisfied at each interior point of the solid, while I(e) must hold on the surface.

II. *Plastic Deformation Theory.*

a) $\sigma_{ij,i} = 0;$

b) $\epsilon_{kk} = 0;$

c) $s_{ij} = 2G_0\epsilon_{ij} - [c_1 + c_2E + c_3E^2 + \cdots]E\epsilon_{ij}$, where $c_n = -2a_{2n+1}/3;$

d) $\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i});$ and

e) $u_i = U_i$.

The possibility of solving these non-linear partial differential equations for σ_{ij} and u_i in closed form is remote for most practically important problems. The procedure to be used in this paper assumes that the solutions may be represented by power series expansions of the form

$$\begin{aligned} \sigma_{ij} &= \sum_{p=0}^{\infty} \sigma_{ij}^{(2p+1)} t^{2p+1}, \\ \epsilon_{ij} &= \sum_{p=0}^{\infty} \epsilon_{ij}^{(2p+1)} t^{2p+1}, \\ u_i &= \sum_{p=0}^{\infty} u_i^{(2p+1)} t^{2p+1}, \end{aligned} \tag{7.1}$$

and

$$U_i = \sum_{p=0}^{\infty} U_i^{(2p+1)} t^{2p+1},$$

where the unknown coefficients of the odd powers of the conveniently chosen parameter t, which may be thought of as time, depend on the space variables only. Substituted in the basic equations of the flow theory these expansions lead to the following relations:

a) $\sigma_{ij,i}^{(1)} + 3\sigma_{ij,i}^{(3)}t^2 + 5\sigma_{ij,i}^{(5)}t^4 + \cdots = 0;$

b) $\epsilon_{kk}^{(1)} + 3\epsilon_{kk}^{(3)}t^2 + 5\epsilon_{kk}^{(5)}t^4 + \cdots = 0;$

c) $s_{ij}^{(1)} + 3s_{ij}^{(3)}t^2 + 5s_{ij}^{(5)}t^4 + \cdots = 2G_0\{\epsilon_{ij}^{(1)} + 3\epsilon_{ij}^{(3)}t^2 + 5\epsilon_{ij}^{(5)}t^4 + \cdots\}$ (7.2)

$$-\frac{4}{3}\Big[b_1+\frac{2}{3}\,b_2(E_{11}t^2+2E_{13}t^4+\cdots)$$

$$+\frac{4}{9}\,b_3(E_{11}^2t^4+4E_{11}E_{13}t^6+\cdots)+\cdots\Big]$$

$$\cdot[E_{11}t+4E_{13}t^3+\cdots][\epsilon_{ij}^{(1)}t+\epsilon_{ij}^{(3)}t^3+\epsilon_{ij}^{(5)}t^5+\cdots],$$

where $E_{rs}=\epsilon_{ij}^{(r)}\epsilon_{ij}^{(s)}$;

d) $\epsilon_{ij}^{(1)}+3\epsilon_{ij}^{(3)}t^2+5\epsilon_{ij}^{(5)}t^4+\cdots=\frac{1}{2}[\{u_{i,j}^{(1)}+3u_{i,j}^{(3)}t^2+5u_{i,j}^{(5)}t^4+\cdots\}$
$+\{u_{j,i}^{(1)}+3u_{j,i}^{(3)}t^2+5u_{j,i}^{(5)}t^4+\cdots\}]$;

and

e) $u_i^{(1)}+3u_i^{(3)}t^2+5u_i^{(5)}t^4+\cdots=U_i^{(1)}+3U_i^{(3)}t^2+5U_i^{(5)}t^4+\cdots$.

The unknown coefficients of the linear terms in t of the expansions (7.1) are the solutions of the set of differential equations obtained for $t=0$ in Eqs. (7.2), namely

a) $\sigma_{ij,i}^{(1)}=0$;

b) $\epsilon_{kk}^{(1)}=0$;

c) $s_{ij}^{(1)}=2G_0\epsilon_{ij}^{(1)}$; (7.3)

d) $\epsilon_{ij}^{(1)}=\frac{1}{2}\{u_{i,j}^{(1)}+u_{j,i}^{(1)}\}$; and

e) $u_i^{(1)}=U_i^{(1)}$.

These equations represent the basic equations for an incompressible elastic body, for which the existence [7, 8] and uniqueness [9, pp. 92, 93] are known. We assume that the linear equations (7.3) can then be integrated either in closed form or by some numerical method to yield $\sigma_{ij}^{(1)}$ and $u_i^{(1)}$. Substituting Eqs. (7.3) into (7.2), factoring out t^2, and letting t tend towards zero, we obtain

a) $\sigma_{ij,i}^{(3)}=0$;

b) $\epsilon_{kk}^{(3)}=0$;

c) $s_{ij}^{(3)}=2G_0\epsilon_{ij}^{(3)}-\frac{4}{9}\,b_1E_{11}\epsilon_{ij}^{(1)}$; (7.4)

d) $\epsilon_{ij}^{(3)}=\frac{1}{2}\{u_{i,j}^{(3)}+u_{j,i}^{(3)}\}$; and

e) $u_i^{(3)}=U_i^{(3)}$.

These equations, too, have the form of the basic equations for an incompressible elastic body. Indeed, they are equivalent to

$$\text{a)}\ \sigma_{ij,i}^{(3)} = \frac{4}{9} b_1 (E_{11}\epsilon_{ij}^{(1)})_{,i}\ ;$$

$$\text{b)}\ \epsilon_{kk}^{(3)} = 0;$$

$$\text{c)}\ s_{ij}^{(3)} = 2G_0\epsilon_{ij}^{(3)}; \tag{7.5}$$

$$\text{d)}\ \epsilon_{ij}^{(3)} = \tfrac{1}{2}\{u_{i,j}^{(3)} + u_{j,i}^{(3)}\}; \qquad \text{and}$$

$$\text{e)}\ u_i^{(3)} = U_i^{(3)},$$

where we introduce a fictitious body force expressed in terms of the solution of the first system of Eqs. (7.3). It is readily seen that all other unknown coefficients in the expansions (7.1) can be calculated from sets of equations similar in structure to Eqs. (7.5).

Carrying out an analogous procedure in the case of the basic equations of the plastic deformation theory we find that the calculation of the unknown coefficients of the linear terms in t of expansions (7.1) reduces, as for the flow theory, to the solving of Eqs. (7.3). The differential equations for $\sigma_{ij}^{(3)}$, $\epsilon_{ij}^{(3)}$, etc. are those of Eqs. (7.4) with the exception that the factor $4b_1/9$ in Eq. (7.4c) is replaced by the factor $2c_1/3$; however, from the definitions,

$$\frac{4}{9} b_1 = \frac{2}{3} c_1 = -\frac{4}{9} a_3 .$$

We thus find that the two theories give solutions which are identical as far as the terms linear and cubic in t are concerned.

The next higher order approximation will produce a difference between the solutions as given by the flow and deformation theories. For the flow theory, we have

$$s_{ij}^{(5)} = 2G_0\epsilon_{ij}^{(5)} + \frac{4}{15} a_3 E_{11}\epsilon_{ij}^{(3)} + \frac{16}{15} a_3 E_{13}\epsilon_{ij}^{(1)} + \frac{8}{27} a_5 E_{11}^2 \epsilon_{ij}^{(1)}, \tag{7.6}$$

and for the deformation theory

$$s_{ij}^{(5)} = 2G_0\epsilon_{ij}^{(5)} + \frac{4}{9} a_3 E_{11}\epsilon_{ij}^{(3)} + \frac{8}{9} a_3 E_{13}\epsilon_{ij}^{(1)} + \frac{8}{27} a_5 E_{11}^2 \epsilon_{ij}^{(1)}. \tag{7.7}$$

Thus, from this stage on, the two theories will differ as a rule.

8. Conditions for similarity. In order that the fifth order terms be the same for both theories, so that the solutions agree up to the terms of fifth order, we have the condition

$$E_{13}\epsilon_{ij}^{(1)} = E_{11}\epsilon_{ij}^{(3)}. \tag{8.1}$$

This equation is satisfied whenever

$$\epsilon_{ij}^{(3)} = f(x, y, z)\epsilon_{ij}^{(1)}, \tag{8.2}$$

where $f(x, y, z)$ is an arbitrary function. However, the strains $\epsilon_{ij}^{(3)}$ as given by (8.2) will satisfy the compatibility equations [9, p. 27] only if the function $f(x, y, z)$ reduces to a constant. Accordingly,

$$\epsilon_{ij}^{(3)} = A\epsilon_{ij}^{(1)}. \tag{8.3}$$

If we define $\epsilon_{ij}/E^{1/2}$ as the normalized tensor of ϵ_{ij} , condition (8.3) states that the normalized tensors $\epsilon_{ij}^{(1)}$ and $\epsilon_{ij}^{(3)}$ must be identical. Substituting this value of $\epsilon_{ij}^{(3)}$ into the stress-strain relations for $s_{ij}^{(7)}$ and adding a further restriction that

$$\epsilon_{ij}^{(5)} = B\epsilon_{ij}^{(1)}, \tag{8.4}$$

we find that now the two theories are identical for seventh order terms. Thus the first four terms of the expansion for s_{ij} , i.e.,

$$s_{ij} = s_{ij}^{(1)}t + s_{ij}^{(3)}t^3 + s_{ij}^{(5)}t^5 + s_{ij}^{(7)}t^7$$

are identical for both theories whenever

$$\epsilon_{ij} = \epsilon_{ij}^{(1)}[t + At^3 + Bt^5] + \epsilon_{ij}^{(7)}t^7. \tag{8.5}$$

Higher order agreement may be obtained by similar restrictions.

It is of interest to determine conditions under which the actual solutions as given by the theories of plastic flow and plastic deformation, and not as approximated by a finite number of terms of expansions (7.1), are identical. Differentiating the plastic deformation equations II with respect to time and comparing with the plastic flow equations I, we find that there is no difference between the theories if

$$2E\epsilon_{ij}^{*} - E^{*}\epsilon_{ij} = 0. \tag{8.6}$$

This condition is fulfilled if

$$\epsilon_{ij} = \epsilon_{ij}^{(1)}(x, y, z)g(t),$$

where $g(t)$ is a monotonically increasing function of t. The two theories thus are seen to furnish identical predictions if the ratios between the various strain components at a generic point remain constant throughout the loading process.

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—NOTES—

A NOTE ON A SOLUTION TO POSSIO'S INTEGRAL EQUATION FOR AN OSCILLATING AIRFOIL IN SUBSONIC FLOW*

By JOHN W. MILES (*University of California at Los Angeles*)

Summary. Following a procedure analogous to that used by Schwarz for solving the integral equation for an oscillating airfoil in incompressible flow, a method of attack on the same problem for subsonic, compressible flow is outlined. The circulation and kernel to the circulation integral equations are expanded in series with the coefficients M^{2n} and $M^{2n} \ln M$, and a corresponding expansion for the Green's function is thereby implicitly determined. The solution up to (but not including) terms of $O(M^4, M^4 \ln M)$ is stated. A method of iteration for improving a given solution is also indicated.

The circulation integral equation. The problem to be considered is that of a thin airfoil located near the x axis between $x = \pm 1$ in an airstream having a subsonic velocity U in the positive x direction. The theory is linearized in the sense that the downwash $w(x, t)$ (positive down) may be specified along the x axis between $x = \pm 1$ and is small compared to U. The function which it is desired to obtain from $w(x, t)$ is the circulation, $\gamma(x, t)$ which is so defined that the pressure jump, $\Pi(x, t)$ at any point on the airfoil is given by the Kutta-Joukowsky law

$$\Pi(x, t) = \rho U \gamma(x, t). \tag{1}$$

For a harmonic time variation of the form $\exp(i\omega t)$, Küssner[1] has shown that $\gamma(x, t)$ and $w(x, t)$ are connected by the integral equation

$$w(x, t) = \frac{1}{2\pi\mu} \int_{-1}^{+1} \gamma(\xi, t) \exp [ik(\xi - x)] R[M, \mu^2 k(x - \xi)](x - \xi)^{-1} \, d\xi, \tag{2}$$

where M is Mach's modulus, μ is the Lorentz contraction factor $(1 - M^2)^{-1/2}$, k is the frequency parameter $\omega b/U$ (b, the semi-chord, is the unit of length used throughout the anslysis), and $R(M, y)$ is given by

$$R(M, y) = \frac{i\pi}{2} My \int_{-\infty}^{y} \exp(iu) H_1^{(2)}(M \mid u \mid) \mid u \mid^{-1} du, \tag{3}$$

$H_1^{(2)}(z)$ being Hankel's cylindrical function of the second kind.[2] The notation has been converted to that generally used in this country; Küssner uses $\beta = M$, $\nu = \omega$, $\omega = ik$.

The incompressible problem. Soehngen[3] has shown that the only solution to

$$g(x) = \frac{1}{2\pi} \int_{-1}^{1} f(\xi)(x - \xi)^{-1} \, d\xi \tag{4}$$

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[1] H. G. Küssner, *Algemeine Tragflächentheorie*, Luftfahrtforschung **17**, 337-354 (1940).

[2] G. N. Watson, *Bessel functions*, Macmillan, New York, 1945.

[3] H. Soehngen, *Die Lösungen der Integralgleichung*

$$g(x) = (2\pi)^{-1} \int_{-a}^{a} (x - \xi)^{-1} f(\xi) \, d\xi$$

und deren Anwendung in der Tragflügeltheorie, Math. Z. **45** (1939).

for which $f(1)$ is finite (in the present problem, the Kutta condition), is given by

$$f(x) = -\frac{2}{\pi}\left(\frac{1-x}{1+x}\right)^{1/2}\int_{-1}^{1} g(\xi)\left(\frac{1+\xi}{1-\xi}\right)^{1/2}(x-\xi)^{-1}\,d\xi. \tag{5}$$

Using this result, Schwarz[4] has given the incompressible ($M = 0$) solution to (1) and the Kutta condition

$$\gamma(1, t) = 0. \tag{6}$$

His solution is

$$\gamma(\cos\theta, t) = \frac{1}{\pi}\int_0^{\pi} G_0(k, \theta, \varphi)w(k, \varphi, t)\,d\varphi, \tag{7}$$

$$x = \cos\theta, \qquad \xi = \cos\varphi, \tag{8}$$

$$\begin{aligned} G_0(k, \theta, \varphi) = {} & ik\ln\left[\frac{1-\cos(\theta+\varphi)}{1-\cos(\theta-\varphi)}\right] + 2(\cos\varphi - \cos\theta)^{-1}\sin\theta \\ & + [(1-\cos\varphi) + T(k)(1+\cos\varphi)]\tan\left(\frac{\theta}{2}\right), \end{aligned} \tag{9}$$

$$T(k) = \frac{-iH_0^{(2)}(k) + H_1^{(2)}(k)}{iH_0^{(2)}(k) + H_1^{(2)}(k)}, \tag{10}$$

where $H_0^{(2)}$ and $H_1^{(2)}$ are Hankel functions in Watson's notation.[2]

Compressible case. In seeking a solution to (2) for $M \neq 0$, it is expedient to introduce the modified functions

$$w^*(x) = \mu\exp(-i\kappa x)w(x), \tag{11}$$

$$\gamma^*(x) = \exp(-i\kappa x)\gamma(x), \tag{12}$$

$$\kappa = \left(\frac{M^2}{1-M^2}\right)k = M^2k^*, \tag{13}$$

$$k^* = \mu^2 k \tag{14}$$

and rewrite the integral Eq. (2) as

$$w^*(x, t) = \int_{-1}^{1} K[M, k(x-\xi)]\gamma^*(\xi, t)\,d\xi, \tag{15}$$

$$K(M, y) = \frac{1}{2\pi}y^{-1}\exp(-iy)R(M, y). \tag{16}$$

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[4]L. Schwarz, *Berechnung der Druckverteilung einer harmonisch sich verformenden Tragfläche in ebener Strömung*, Luftfahrtforschung **17**, 379-386 (1940).

Expanding the Hankel function in (3), the kernel in (16) may be written

$$K(M, y) = \frac{1}{2\pi} \exp(-iy) \int_{-\infty}^{u} \exp(iu) \Bigg\{ -u^{-2} + \sum_{m=0}^{\infty} \frac{(-)^m M^{2m+2} u^{2m}}{2^{2m} m!(m+1)!} \left[\ln\left(M \left| \frac{u}{2} \right|\right) + C - \frac{1}{2(m+1)} - \sum_{n=1}^{m} \left(\frac{1}{n}\right) + \frac{i\pi}{2} \right] \Bigg\} du \tag{17}$$

where C is Euler's constant. It will generally be convenient to integrate each of the terms retained in the integrand repeatedly by parts until the only integral remaining in the kernel is $\int_{-\infty}^{u} u^{-1} \exp(iu)\, du$.

A solution (15) can be carried out by writing

$$K(M, y) = \sum_{n=0}^{\infty} M^{2n} K_{2n}(y) + \ln(M) \sum_{n=1}^{\infty} M^{2n} K'_{2n}, \tag{18}$$

$$\gamma^*(x) = \sum_{n=0}^{\infty} M^{2n} \gamma_{2n}(x) + \ln(M) \sum_{n=1}^{\infty} M^{2n} \gamma'_{2n}(x) \tag{19}$$

and proceeding, in a manner entirely analogous to that utilized by Schwarz, towards a result of the form

$$\gamma(\theta) = \frac{\mu}{\pi} \int_0^{\pi} \exp[i\kappa(\cos\theta - \cos\varphi)] G(k^*, \theta, \varphi) w(k, \varphi, t)\, d\varphi, \tag{20}$$

$$G(k^*, \theta, \varphi) = \sum_{n=0}^{\infty} M^{2n} G_{2n}(k^*, \theta, \varphi) + \ln(M) \sum_{n=1}^{\infty} M^{2n} G'_{2n}(k^*, \theta, \varphi). \tag{21}$$

The algebraic manipulations required to solve for the G_{2n} and G'_{2n} are rather lengthy, and it is probably not feasible to include more than a few terms. However, the solution can be improved by iteration. Thus, if $\gamma^{*(N)}$ denotes the solution to

$$w^* = \int_{-1}^{1} K^{(N)} \gamma^{*(N)}\, d\xi, \tag{22}$$

the remainder $[\gamma^* - \gamma^{*(N)}]$ is given by the solution to

$$w^{*(N)} = \int_{-1}^{1} [K^{(N)} - K] \gamma^{*(N)}\, d\xi = \int_{-1}^{1} K[\gamma^* - \gamma^{*(N)}]\, d\xi, \tag{23}$$

where the left side to (23) may be regarded as a new velocity distribution $w^{*(N)}$.

The author has carried out the solution for $N = 1^*$. In this case G_0 is given by (9) (noting that k must be replaced by k^*), while G_2 and G'_2 are given by

$$G_2(k^*, \theta, \varphi) = k^* \tan\left(\frac{\theta}{2}\right)(1 + \cos\varphi) \Bigg\{ \left[i\left(\alpha + \frac{1}{4}\right) - \frac{1}{2k^*} \right] \left(\frac{1+T}{2}\right)^2 + \left(\frac{1}{2k^*} - i\right)\left(\frac{1+T}{2}\right) + \frac{i}{4} \Bigg\} - k^* \left(\frac{1+T}{2}\right) \tan\left(\frac{\theta}{2}\right) \sin^2\varphi - k^{*2} \sin\theta \left[\alpha\left(\frac{1+T}{2}\right)(1 + \cos\varphi) + ik^*\left(\alpha - \frac{1}{2}\right) \sin^2\varphi \right]$$

*Details of the solution are available from the author.

$$+ \frac{k^{*2}}{4}\left[1 - \left(\frac{1+T}{2}\right)\right]\sin\theta\cos\theta$$

$$- k^{*2}\sin\varphi(\cos\theta - \cos\varphi)\left[1 + \frac{ik^*}{2}(\cos\theta - \cos\varphi)\right]\ln\left[\frac{1-\cos(\theta+\varphi)}{1-\cos(\theta-\varphi)}\right], \quad (24)$$

$$G_2'(k^*, \theta, \varphi) = \frac{k^*}{2}\left(\frac{1+T}{2}\right)\tan\left(\frac{\theta}{2}\right)\left[i\left(\frac{1+T}{2}\right)(1+\cos\varphi) - k^*\sin^2\varphi\right]$$

$$- \frac{k^{*2}}{2}\left(\frac{1+T}{2}\right)\sin\theta\left[\left(\frac{1+T}{2}\right)(1+\cos\varphi) + ik^*\sin^2\varphi\right], \quad (25)$$

$$\alpha = \left(\frac{2C - 1 + i\pi}{4}\right) + \frac{1}{2}\ln\left(\frac{k}{4}\right). \quad (26)$$

The function $[(1 + T(k^*))/2]$ is the $C(k^*)$ used by Theodorsen.[5]

Other solutions. Other solutions of (2) have been given by Possio[6,7], Schade[8], Dietze[9] and Eichler[10]. The only extensive numerical results available are those of Possio for the lift and moment on a flat plate airfoil due to plunging and pitching motions. The collocation technique used by Possio is probably not practical for control surface motion.

Due to the large expenditure of labor which will be required to calculate subsonic flutter derivatives over (usefully) complete ranges of M and k, it appears that a preliminary study of all available methods should be conducted, probably under the auspices of one of the government agencies. While it appears to the author that the work of Schade[8] is probably best suited to a large scale computing program, some calculations should probably be made using each of the methods mentioned.

Further discussion of the German work has been given by Biot, et al.[11]

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THE NORMAL EQUATIONS OF THE METHOD OF LEAST SQUARES AND THEIR SOLUTION*

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The literature of the numerical solution of linear equations is extensive.** The fact that war work forced many mathematicians into contact with numerical work has brought with it a better insight into the mathematical problems involved. In 1947, von Neumann and Goldstine[1] discussed an analysis of the accuracy of the best-known methods of solving linear equations and an analysis of the steps involved. The present paper provides the applied mathematician with a method by which he can improve the accuracy of his solution, step by step. At the same time, an attempt is made to guide the computer by transforming the problem into one of analytic geometry.

The problem of solving n linear equations with n unknowns is equivalent to finding the components of a vector $\mathbf{y}$ with respect to n independent vectors $\mathbf{a}_i$:

$$\xi_i \mathbf{a}_i = \mathbf{y}. \tag{1}$$

Multiplication by $\mathbf{a}_j$ gives the equations

$$\xi_i \mathbf{a}_i \cdot \mathbf{a}_j = \mathbf{y} \cdot \mathbf{a}_j , \tag{2}$$

having a symmetric determinant.

The normal equations of the method of least squares can be written in the same form. The geometrical problem is as follows:

Given in $n > k$-dimensional space the k vectors $\mathbf{a}_1 , \cdots , \mathbf{a}_k$ and a vector $\mathbf{y}$, we want to determine a vector $\mathbf{d} = \xi_i \mathbf{a}_i$ such that

$$(\mathbf{y} - \mathbf{d})^2 = \text{min.}, \tag{3}$$

i.e., we want the best representation of $\mathbf{y}$ as a linear manifold of $\mathbf{a}_i$.

The solution is obviously the vector $\mathbf{d}$, which is the projection of $\mathbf{y}$ into the manifold $\mathbf{a}_i$ so that

$$(\mathbf{y} - \mathbf{d}) \cdot \mathbf{a}_i = 0, \tag{4}$$

a result which also can be obtained by differentiating Eq. (3) with respect to $\mathbf{a}_i$. Inserting into Eq. (4) the value of $\mathbf{d}$, we obtain

$$\xi_i \mathbf{a}_i \cdot \mathbf{a}_j = \mathbf{y} \cdot \mathbf{a}_j , \tag{5}$$

the normal equations of the problem of least squares. These equations are identical with Eqs. (2), the only difference being that the $\mathbf{a}_i$ are vectors in a space of higher dimensions.

Equations (5) are the normal equations of the problem of least squares. Their solution, in general, presents a difficulty if, and only if, the determinant of $\mathbf{a}_i$ is small, that means, if the vectors $\mathbf{a}_i$ are "nearly linearly dependent."

In an earlier article[5] the replacement of the vectors $\mathbf{a}_i$ by unit vectors was suggested:

$$\mathbf{e}_i = \mathbf{a}_i / (\mathbf{a}_i^2)^{1/2}; \qquad \mathbf{f} = \mathbf{y} / (\mathbf{y}^2)^{1/2}. \tag{6}$$

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**Comprehensive summaries have been given by Hotelling[2], Dwyer,[3] and Bodewig.[4]

In this case,

$$\mathbf{e}_i^2 = \mathbf{f}^2 = 1$$
$$\mathbf{e}_i \cdot \mathbf{e}_j = \cos(\mathbf{e}_i, \mathbf{e}_j), \tag{7}$$

i.e., the matrix of Eqs. (5) has all its diagonal elements equal to unity and the other elements smaller than unity.

We now re-order $\mathbf{e}_1, \cdots, \mathbf{e}_k$ with the aim of choosing e_1 and e_2 such that $[e_1e_2]^2$ has the largest possible value, choosing e_3 such that $[e_1e_2e_3]^2$ has the largest possible value, and so on.*

For practical reasons, this criterion may be replaced by the following, which, although not rigorous, is easier to apply. We choose e_1, e_2 such that $(e_1e_2)^2$ has the smallest value, then add e_3 such that $(e_2e_3)^2 + (e_1e_3)^2$ has the smallest value, and so on. These values form, on account of Hadamard's famous theorem, a majorant of the values $[e_1e_2 \cdots e_i]^2$. The re-ordering is done to investigate whether a projection into a space of a smaller number of variables gives a sufficiently close approximation. We must keep in mind that in practical problems the data are only known to a certain number of digits. Therefore, it is only necessary to determine a solution so as to reproduce the accuracy of the data.

To solve the normal equations, two procedures seem possible: one is to change the right-hand side, i.e., to replace the vectors $\mathbf{y}$ successively by smaller vectors; the other is to replace the vectors $\mathbf{a}_i$ on the left-hand side by a series of other vectors which permit the solution of the normal equations. We shall make suggestions with respect to both methods. The determinant of the normal equations is the discriminant of the quadratic form,

$$\left(\sum_1^k \mathbf{a}_i\xi_i\right)^2 = \sum_1^k \mathbf{a}_i\mathbf{a}_j\xi_i\xi_j\,, \tag{8}$$

and as such is positive and definite. Since it is derived from a finite number of digits in machine computation, however, the number of significant figures in the calculated value of the determinant may be small. In either case, the homogeneous equations (right-hand side equal to zero) have only the solution: $\xi_1 = \cdots = \xi_k = 0$.

Let us assume that we have given a vector $\mathbf{b}$ which might be an approximation to $\mathbf{y}$. Such a vector $\mathbf{b}$ permits us to obtain a new and smaller value for the right-hand side of Eqs. (5). If we introduce

$$\mathbf{b} = \beta_i\mathbf{a}_i \tag{9}$$

into Eqs. (5), we obtain

$$\xi_i\mathbf{a}_i\mathbf{a}_j = (\mathbf{y} - \mathbf{b})\mathbf{a}_j = \mathbf{y}\mathbf{a}_j - \beta_i\mathbf{a}_i\mathbf{a}_j\,. \tag{10}$$

The sum of the squares for the new vector is given by

$$(\mathbf{y} - \beta_i\mathbf{a}_i)^2 = \mathbf{y}^2 - 2\beta_i\mathbf{y}\mathbf{a}_i + \beta_i\beta_j\mathbf{a}_i\mathbf{a}_j\,. \tag{11}$$

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*The symbol $[e_1e_2 \cdots e_\rho]^2$ is an abbreviation for the determinant of the products e_ie_j for $i, j = 1, \cdots, \rho$, taken from Grassmann's symbolic notation. It is the square of the volume of the parallelepiped formed by the vectors $e_1, \cdots, e_\rho$.

The reader may notice that the computation of the coefficients in (10) and (11) presupposes only a knowledge of the β, the $\mathbf{a}_i\mathbf{a}_j$, and the $\mathbf{ba}_i$, i.e., the data of the $k+1$ by k matrix. It is not necessary to go back to the original vectors in n-dimensional space.

If a number of approximation vectors $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_m$ are given, then it is possible to compute the best multiple of them to be taken from the right-hand side of the equation. The best approximation is given by

$$\mathbf{B} = \eta_i \mathbf{b}_i \tag{12}$$

with

$$\eta_j \mathbf{b}_j \mathbf{b}_j = \mathbf{b}_i \mathbf{y}.$$

The reader may notice again that the expressions in Eq. (12) can be computed from the coefficients of $\mathbf{b}_i$ with respect to $\mathbf{a}_i$ without having recourse to n-dimensional space.

Of much greater importance for the solution of the normal equation is the transformation of the left-hand side.

There are two different transformations of the left-hand side of the equation which have played a role in the literature of the problem. We shall analyze these transformations in the remainder of the paper. The first method will be called "the improved Gauss-Doolittle"[9] method. The second one is the square-root method found independently by Schur,[6] Banachiewicz,[7] and Dwyer.[3] We shall study both methods theoretically for the general case in which the vectors are not reduced, but we recommend the second method particularly, for the reduction to unit vectors.

Strangely enough, the best way to discuss these methods seems not to be found in the literature.* Both methods can be described as using orthogonal vectors. Let us discuss first the Gauss-Doolittle method. We find a system of orthogonal vectors $\mathbf{o}_1, \cdots, \mathbf{o}_k$ such that $\mathbf{o}_1 = \mathbf{a}_1$, that $\mathbf{o}_2$ lies in the plane of $\mathbf{a}_1$ and $\mathbf{a}_2$, that $\mathbf{o}_3$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and such that in all cases

$$[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_i]^2 = [\mathbf{o}_1\mathbf{o}_2\mathbf{o}_i]^2. \tag{13}$$

This leads to

$$\begin{aligned} \mathbf{a}_1 &= \mathbf{o}_1, \\ \mathbf{a}_2 &= \delta_{12}\mathbf{o}_1 + \mathbf{o}_2, \\ \mathbf{a}_3 &= \delta_{13}\mathbf{o}_1 + \delta_{23}\mathbf{o}_2 + \mathbf{o}_3, \\ &\cdots \end{aligned} \tag{14}$$

The matrix Δ of the δ_{ij} is a triangular matrix which we can calculate as follows. From Eqs. (14) we find that

$$\begin{aligned} \mathbf{a}_1^2 &= \mathbf{o}_1^2, & \mathbf{a}_1\mathbf{a}_j &= \delta_{1j}\mathbf{o}_1^2, \\ \mathbf{a}_2^2 &= \delta_{12}\mathbf{o}_1^2 + \mathbf{o}_2^2, & \mathbf{a}_2\mathbf{a}_j &= \delta_{12}\delta_{1j}\mathbf{o}_1^2 + \delta_{2j}\delta_2^2, \\ \mathbf{a}_3^2 &= \delta_{13}\mathbf{o}_1^2 + \delta_{23}\mathbf{o}_2^2 + \mathbf{o}_3^2, & \mathbf{a}_3\mathbf{a}_j &= \delta_{13}\delta_{1j}\mathbf{o}_1^2 + \delta_{23}\delta_{2j}\mathbf{o}_2^2 + \delta_{3j}\mathbf{o}_3^2. \end{aligned} \tag{15}$$

*As far as we know, Kolmogoroff[10] is the only other author who has used vector notation for the least-squares method. His paper, published shortly before our first communication,[5] reached us only recently.

If we abbreviate the *diagonal matrix*,

$$\begin{pmatrix} \mathfrak{o}_1^2 & 0 & 0 & 0 & 0 \\ 0 & \mathfrak{o}_2^2 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \mathfrak{o}_k^2 \end{pmatrix} = O \tag{16}$$

these equations can be written in the matrix form

$$A = \Delta' O \Delta. \tag{17}$$

Since $\mathfrak{a}_i$ and $\mathfrak{a}_j$ are given, the computation of δ_{ij} can now be done in the following way. Equations (15) give the relations

$$\begin{aligned} \mathfrak{o}_1^2 &= \mathfrak{a}_1^2 , & \delta_{1j} &= (\mathfrak{a}_1\mathfrak{a}_j)/\mathfrak{o}_1^2 , \\ \mathfrak{o}_2^2 &= \mathfrak{a}_2^2 - \delta_{12}^2\mathfrak{o}_1^2 , & \delta_{2j} &= (\mathfrak{a}_2\mathfrak{a}_j - \delta_{12}\delta_{1j}\mathfrak{o}_1^2)/\mathfrak{o}_2^2 , \\ \mathfrak{o}_3^2 &= \mathfrak{a}_3^2 - \delta_{13}^2\mathfrak{o}_1^2 - \delta_{23}^2\mathfrak{o}_2^2 , & \delta_{3j} &= (\mathfrak{a}_3\mathfrak{a}_j - \delta_{13}\delta_{1j}\mathfrak{o}_1^2 - \delta_{23}\delta_{2j}\mathfrak{o}_2^2)/\mathfrak{o}_3^2 . \end{aligned} \tag{18}$$

Having obtained $\mathfrak{o}_i^2$ and the triangular matrix, δ_{ik}, we can solve the least-squares equations in two simple steps. The equations $A\xi_j = \eta_i$ are equivalent to $\Delta' O \Delta \xi_j = \eta_i$. We write

$$\Delta\xi_j = \zeta_\rho \tag{19}$$

and

$$O\Delta'\zeta_\rho = \eta_i . \tag{20}$$

These equations are easily solved because of the triangularity of the matrix Δ. Written in full, Eqs. (20) are

$$\begin{aligned} \mathfrak{o}_1^2\zeta_1 &= \eta_1 , \\ \mathfrak{o}_1^2\delta_{12}\zeta_1 + \mathfrak{o}_2^2\zeta_2 &= \eta_2 , \\ \mathfrak{o}_1^2\delta_{13}\zeta_1 + \mathfrak{o}_2^2\delta_{23}\zeta_2 + \mathfrak{o}_3^2\zeta_3 &= \eta_3 , \end{aligned} \tag{21}$$

or, solved

$$\begin{aligned} \zeta_1 &= \eta_1/\mathfrak{o}_1^2 , \\ \zeta_2 &= (\eta_2 - \mathfrak{o}_1^2\delta_{12}\zeta_1)/\mathfrak{o}_2^2 , \\ \zeta_3 &= (\eta_3 - \mathfrak{o}_1^2\delta_{13}\zeta_1 - \mathfrak{o}_2^2\delta_{23}\zeta_2)/\mathfrak{o}_3^2 . \end{aligned} \tag{22}$$

The computation of ξ_i is then obtained by the equations

$$\begin{aligned} \xi_k &= \zeta_k , \\ \xi_{k-1} &= \zeta_{k-1} - \delta_{k,k+1}\xi_k . \end{aligned} \tag{23}$$

It may be noted that the ζ_i are the coefficients of the vector $\mathbf{x} = \xi_i \mathbf{a}_i$ with respect to $\mathbf{o}_i$, for

$$\mathbf{x} = \xi_i \mathbf{a}_i = \xi_i \delta_{ki} \mathbf{o}_i = \zeta_k \mathbf{o}_i \,. \tag{24}$$

The discussion of the *square-root method* is similar, except that, instead of choosing orthogonal vectors which fulfill Eq. (13), we choose orthogonal unit vectors $\mathbf{o}_\nu$ in the direction of $\mathbf{o}_i$. This leads again to a triangular matrix of coefficients:

$$\begin{aligned} \mathbf{a}_1 &= \gamma_{11}\mathbf{o}_1 \,, \\ \mathbf{a}_2 &= \gamma_{12}\mathbf{o}_1 + \gamma_{22}\mathbf{o}_2 \,, \\ \mathbf{a}_3 &= \gamma_{13}\mathbf{o}_1 + \gamma_{23}\mathbf{o}_2 + \gamma_{33}\mathbf{o}_3 \,, \end{aligned} \tag{25}$$

with the distinction that

$$A = \Gamma'\Gamma, \tag{26}$$

since

$$\mathbf{a}_i \mathbf{a}_j = \gamma_{\nu i}\gamma_{\nu j} \,. \tag{27}$$

Hence, by this procedure the original matrix becomes the product of two triangular matrices.

The computation of γ_{ij} proceeds similarly to that of δ_{ij}. Equations (25) give

$$\begin{aligned} \gamma_{11}^2 &= \mathbf{a}_1^2 \,, & \gamma_{11}\gamma_{1j} &= \mathbf{a}_1\mathbf{a}_j \,, \\ \gamma_{12}^2 + \gamma_{22}^2 &= \mathbf{a}_2^2 \,, & \gamma_{12}\gamma_{1j} + \gamma_{22}\gamma_{2j} &= \mathbf{a}_2\mathbf{a}_j \,. \end{aligned} \tag{28}$$

Thus, we find that

$$\begin{aligned} \gamma_{11} &= (\mathbf{a}_1^2)^{1/2}, & \gamma_{1j}\gamma_{11} &= \mathbf{a}_1\mathbf{a}_j \,, \\ \gamma_{22} &= (\mathbf{a}_2^2 - \gamma_{12}^2)^{1/2}, & \gamma_{2j}\gamma_{22} &= \mathbf{a}_2\mathbf{a}_j - \gamma_{1j}\gamma_{12} \,, \\ \gamma_{33} &= (\mathbf{a}_3^2 - \gamma_{13}^2 - \gamma_{23}^2)^{1/2}, & \gamma_{3j}\gamma_{33} &= \mathbf{a}_3\mathbf{a}_j - \gamma_{2j}\gamma_{22} - \gamma_{1j}\gamma_{11} \,. \end{aligned} \tag{29}$$

The computation of square roots on a calculating machine can be reduced to simple division if an approximate value is known, because, for any value x and a small increase dx, we have

$$x^2/(x + dx) = x - dx. \tag{30}$$

Having found the value of γ_{ij}, we can proceed with the solution of Eqs. (2), again in two steps:

$$\Gamma'\Gamma\xi_j = A\xi_j = \eta_j \tag{31}$$

which splits into

$$\Gamma\xi_j = \tilde{\zeta}_\rho$$

and (32)

$$\Gamma'\tilde{\zeta}_\rho = \eta_i \,.$$

Again we find that because of Eqs. (25) and (32),

$$\mathbf{x} = \xi_i \mathbf{a}_i = \zeta_i \mathbf{o}_i , \tag{33}$$

so that the ζ are components of $\mathbf{x}$ with respect to $\mathbf{o}_i$.

The advantage of the second method in connection with making the vectors $\mathbf{a}$ and $\mathbf{y}$ unit vectors is that all the numbers calculated, with the exception of the final ξ_i , are projections of vectors of unit length into a Cartesian coordinate system; they, therefore, are all smaller than one. The accuracy of all the operations can hence be preserved to the last significant figure, since division does not lead to a loss of significant figures.

A check consists in forming

$$\Gamma'\Gamma = A \tag{34}$$

and making sure that the elements of A are reproduced to the last decimal.

These remarks hold up to and including the computation of the ζ. If the values of ξ_i , when inserted into Eqs. (5), leave small residuals, a correction can be obtained by repeating the solution of Eqs. (5) with the residuals at the right-hand side. A small number of digits will be sufficient to carry through the computation in this case.

The author suggests the following procedure in solving a system of linear equations having a small determinant and a large number of variables. The first step is to change the vectors $\mathbf{a}_i$ and $\mathbf{b}_i$ into unit vectors and to rearrange the vectors in the manner described previously. We then solve part of the equations according to the square-root method, until $\gamma_{\rho\rho}$ has sufficient significant digits, using the number of decimals which the calculating machine permits. This gives a solution for $\rho < k$ unknowns and reduces the first ρ numbers on the right-hand side to zero or nearly zero. These values are put into the remaining $k - \rho$ equations, part of which are solved, and the same procedure is followed until values of all the unknowns are determined. If the sum of the squares of the residuals is small enough, we stop. If not, we repeat the procedure.

Going through it for a second time, we look for the best multiple of the last solution to diminish further the sum of the squares. The procedure is repeated until the sum of the squares becomes compatible with the error of the original data.

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ON REISSNER'S THEORY OF BENDING OF ELASTIC PLATES*

By A. E. GREEN (*Durham, England*)

1. Introduction. The classical theory of bending of elastic plates has recently been extended and improved by Reissner.[1,2,3] His theory takes into account the transverse-shear deformations of the plate and the equations of the theory are obtained by an application of Castigliano's theorem of minimum energy. The object of the present note is to show that Reissner's equations can be obtained directly from the stress equations of equilibrium and the stress-strain relations. Moreover, by consistent use of complex variable notation, the form of the results is simplified. The equations are first obtained for an isotropic material and are then extended to an aeolotropic material which is transversely isotropic in planes parallel to the faces of the plates.

2. Fundamental equations for isotropic plates. Consider Cartesian coordinates x, y, Z and let $z = x + iy$ denote the complex variable with $\bar{z} = x - iy$ the complex conjugate of z. Stresses connected with the coordinate Z are denoted by τ_{zz} , τ_{yz} , σ_z since there is no need to confuse the z in this notation with the complex variable. Attention is directed to stresses in plates bounded by the planes $Z = \pm h$.

When body forces are absent, Stevenson[4] has shown that the stress equations of equilibrium can be expressed in the form

$$\frac{\partial \Phi}{\partial z} + \frac{\partial \Theta}{\partial \bar{z}} + \frac{\partial \Psi}{\partial Z} = 0, \tag{1a}$$

$$\frac{\partial \Psi}{\partial z} + \frac{\partial \bar{\Psi}}{\partial \bar{z}} + \frac{\partial \sigma_z}{\partial Z} = 0, \tag{1b}$$

where a bar placed over a quantity denotes the complex conjugate of that quantity and where

$$\Theta = \sigma_x + \sigma_y , \qquad \Phi = \sigma_x - \sigma_y + 2i\tau_{xy} , \qquad \Psi = \tau_{zz} + i\tau_{yz} . \tag{2}$$

If u, v, w denote the Cartesian components of displacement and if $D = u + iv$, the complex form of the stress-strain relations is

$$(1 - 2\eta)\Theta = 2\mu\left\{\nabla_1^2(F + \bar{F}) + 2\eta \frac{\partial w}{\partial Z}\right\}, \tag{3a}$$

*Received Aug. 13, 1948.

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[2] E. Reissner, J. Appl. Mech. **12**, A68-A77 (1945).

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$$(1 - 2\eta)\sigma_z = 2\mu\left\{\eta\nabla_1^2(F + \bar{F}) + (1 - \eta)\frac{\partial w}{\partial Z}\right\}, \tag{3b}$$

$$\Phi = 16\mu\frac{\partial^2 F}{\partial \bar{z}^2}, \tag{3c}$$

$$\Psi = 2\mu\frac{\partial}{\partial \bar{z}}\left(2\frac{\partial F}{\partial Z} + w\right), \tag{3d}$$

where

$$D = 4\frac{\partial F}{\partial \bar{z}}, \qquad \nabla_1^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{3e}$$

and where Poisson's ratio η and Young's modulus E are related to Lame's constants by

$$\eta = \frac{\lambda}{2(\lambda + \mu)}, \qquad \mu = \frac{E}{2(1 + \eta)}. \tag{4}$$

3. Change of axes. Stevenson has pointed out that the stress combinations Θ, Φ, Ψ are particularly suitable to the problem of transformation of stress and his main results are recorded here. Thus, if

$$\Theta' = \sigma_n + \sigma_s, \qquad \Phi' = \sigma_n - \sigma_s + 2i\tau_{ns}, \qquad \Psi' = \tau_{nz} + i\tau_{sz}, \tag{5a}$$

where

$$z' = n + is = ze^{-i\alpha}, \tag{5b}$$

then

$$\Theta' = \Theta, \qquad \Phi' = e^{-2i\alpha}\Phi, \qquad \Psi' = e^{-i\alpha}\Psi, \qquad D' = e^{-i\alpha}D. \tag{5c}$$

4. Formulation of the problem. It is assumed that the faces of the plate are free from applied shear stresses so that

$$\Psi = 0 \qquad (Z = \pm h), \tag{6a}$$

while the normal traction on the faces is such that

$$\sigma_z = \mp\tfrac{1}{2}p \qquad (Z = \pm h), \tag{6b}$$

where p is a given function of x and y.

Stress-resultants and stress-couples are defined by

$$M_x = \int_{-h}^{h} Z\sigma_x\,dZ, \qquad M_y = \int_{-h}^{h} Z\sigma_y\,dZ, \qquad H = \int_{-h}^{h} Z\tau_{xy}\,dZ,$$
$$V_x = \int_{-h}^{h} \tau_{xz}\,dZ, \qquad V_y = \int_{-h}^{h} \tau_{yz}\,dZ, \tag{7a}$$

or, in complex form,*

*These definitions are different from those given by Stevenson [4].

$$\Lambda = M_x + M_y = \int_{-h}^{h} Z\Theta \, dZ,$$

$$\Gamma = M_x - M_y + 2iH = \int_{-h}^{h} Z\Phi \, dZ, \tag{7b}$$

$$\Psi_0 = V_x + iV_y = \int_{-h}^{h} \Psi \, dZ.$$

By using the boundary conditions (6a) and (6b) and Eqs. (1a) and (1b) it is found that

$$\frac{\partial \Gamma}{\partial z} + \frac{\partial \Lambda}{\partial \bar{z}} - \Psi_0 = 0,$$

$$\frac{\partial \Psi_0}{\partial z} + \frac{\partial \bar{\Psi}_0}{\partial \bar{z}} - p = 0. \tag{8}$$

The boundary conditions (6b) may be satisfied by taking

$$\sigma_z = -\frac{p}{4}\left\{3\left(\frac{Z}{h}\right) - \left(\frac{Z}{h}\right)^3\right\}, \tag{9a}$$

and then, from (1b),

$$\Psi = \frac{3}{4h}\left\{1 - \left(\frac{Z}{h}\right)^2\right\}\Psi_0 , \tag{9b}$$

which gives zero shear stresses at the faces of the plate $Z = \pm h$.

Weighted displacements are now defined by *

$$D^* = 4\frac{\partial F^*}{\partial \bar{z}}, \qquad F^* = \frac{3}{2h^3}\int_{-h}^{h} ZF \, dZ,$$

$$w^* = \frac{3}{4h}\int_{-h}^{h}\left\{1 - \left(\frac{Z}{h}\right)^2\right\} w \, dZ. \tag{10}$$

Relations between these weighted displacements and the stress-resultants and stress-couples can be found by multiplying Eqs. (3a) to (3d) by suitable functions of Z and integrating with respect to Z through the thickness of the plate. Before carrying out this process it is necessary to eliminate $\partial w/\partial Z$ from Eqs. (3a) and (3b) in order to avoid introducing further unknown quantities. Thus Eqs. (3a) and (3b) are replaced by

$$(1 - \eta)\Theta - 2\eta\sigma_z = 2\mu(1 + \eta)\nabla_1^2(F + \bar{F}), \tag{11}$$

and the remaining equation which involves $\partial w/\partial Z$ is not used. Hence

$$\Psi_0 = \frac{10\mu h}{3}\frac{\partial}{\partial \bar{z}}(w^* + 2F^*), \tag{12a}$$

*The choice of a weighting factor Z for the displacement D is natural. Equation (3d) then shows that w must be weighted with a quadratic factor in Z which vanishes at $Z = \pm h$, in order not to introduce further unknown quantities involving values of F at the faces of the plate.

$$\Gamma = \frac{32\mu h^3}{3}\frac{\partial^2 F^*}{\partial \bar{z}^2}, \tag{12b}$$

$$(1-\eta)\Lambda = -\frac{4}{5}h^2\eta p + \frac{4\mu h^3}{3}(1+\eta)\nabla_1^2(F^* + \bar{F}^*). \tag{12c}$$

Equations (8) are now satisfied if

$$\Lambda + \frac{8h^3\mu}{3}\nabla_1^2 F^* = \frac{10\mu h}{3}(w^* + 2F^*), \tag{13a}$$

$$\nabla_1^2(w^* + F^* + \bar{F}^*) = \frac{3p}{5\mu h}. \tag{13b}$$

5. Solution of the differential equations. If

$$F^* = \phi + i\psi, \tag{14}$$

then, from (13b),

$$w^* + 2\phi = \frac{h^2}{5\mu}\left\{\Omega'(z) + \bar{\Omega}'(\bar{z})\right\} + \frac{3}{5\mu h}\nabla_1^2 P, \tag{15}$$

provided that the applied pressure p can be expressed in the form

$$p = \nabla_1^4 P. \tag{16}$$

Also, from (13a),

$$\psi - \frac{2h^2}{5}\nabla_1^2\psi = 0, \tag{17a}$$

$$\Lambda + \frac{8h^3\mu}{3}\nabla_1^2\phi = \frac{10\mu h}{3}(w^* + 2\phi), \tag{17b}$$

and, from (12c),

$$(1-\eta)\Lambda + \frac{4}{5}h^2\eta p = \frac{8\mu h^3}{3}(1+\eta)\nabla_1^2\phi. \tag{17c}$$

With the help of (15), (17b) and (17c), a simple calculation now gives

$$\phi = \frac{1-\eta}{32\mu}\left\{\bar{z}\Omega(z) + z\bar{\Omega}(\bar{z}) + \omega(z) + \bar{\omega}(\bar{z})\right\} + \frac{3(1-\eta)P}{8h^3\mu} + \frac{3\eta\nabla_1^2 P}{20h\mu}, \tag{18a}$$

$$w^* = -\frac{1-\eta}{16\mu}\left\{\bar{z}\Omega(z) + z\bar{\Omega}(\bar{z}) + \omega(z) + \bar{\omega}(\bar{z})\right\} + \frac{h^2}{5\mu}\left\{\Omega'(z) + \bar{\Omega}'(\bar{z})\right\} - \frac{3(1-\eta)P}{4h^3\mu} + \frac{3(2-\eta)\nabla_1^2 P}{10h\mu}, \tag{18b}$$

$$\Lambda = \frac{(1+\eta)h^3}{3}\left\{\Omega'(z) + \bar{\Omega}'(\bar{z})\right\} - \frac{2h^2\eta p}{5} + (1+\eta)\nabla_1^2 P, \tag{18c}$$

$$\Gamma = \frac{32h^3\mu}{3}\frac{\partial^2}{\partial\bar{z}^2}(\phi + i\psi), \tag{18d}$$

$$\Psi_0 = \frac{10h\mu}{3}\frac{\partial}{\partial\bar{z}}(w^* + 2\phi + 2i\psi), \tag{18e}$$

$$D^* = 4\frac{\partial}{\partial\bar{z}}(\phi + i\psi). \tag{18f}$$

The functions $\Omega(z)$, $\omega(z)$ are arbitrary functions of z. These equations for the weighted displacements and for the stress-resultants and stress-couples are in a compact and convenient form for applications. With them are used formulas for the transformation of stress-resultants and stress-couples, which are easily found by integrating Eqs. (5c) with respect to Z, after multiplication by a suitable function of Z.

6. Results for certain aeolotropic materials. Only small changes are needed in the theory when the material of the plate is aeolotropic but is transversely isotropic in planes parallel to the faces of the plate. The stress-strain relations (3a) to (3d) are replaced by

$$\Theta = (c_{11} + c_{12})\nabla_1^2(F + \bar{F}) + 2c_{13}\frac{\partial w}{\partial Z}, \tag{19a}$$

$$\sigma_z = c_{13}\nabla_1^2(F + \bar{F}) + c_{33}\frac{\partial w}{\partial Z}, \tag{19b}$$

$$\Phi = 8(c_{11} - c_{12})\frac{\partial^2 F}{\partial\bar{z}^2}, \tag{19c}$$

$$\Psi = 2c_{44}\frac{\partial}{\partial\bar{z}}\left(2\frac{\partial F}{\partial Z} + w\right). \tag{19d}$$

Since the details of the analysis are similar to those for an isotropic material only the final results are given here. Thus

$$\phi = \frac{c_{33}}{16(c_{11}c_{33} - c_{13}^2)}\left\{\bar{z}\Omega(z) + z\bar{\Omega}(\bar{z}) + \omega(z) + \bar{\omega}(\bar{z})\right\} + \frac{3c_{13}\nabla_1^2 P}{10h(c_{11}c_{33} - c_{13}^2)} + \frac{3c_{33}P}{4h^3(c_{11}c_{33} - c_{13}^2)}, \tag{20a}$$

$$w^* = -\frac{c_{33}}{8(c_{11}c_{33} - c_{13}^2)}\left\{\bar{z}\Omega(z) + z\bar{\Omega}(\bar{z}) + \omega(z) + \bar{\omega}(\bar{z})\right\} + \frac{h^2}{5c_{44}}\left\{\Omega'(z) + \bar{\Omega}'(\bar{z})\right\} - \frac{3c_{33}P}{2h^3(c_{11}c_{33} - c_{13}^2)} + \frac{3}{5h}\left\{\frac{1}{c_{44}} - \frac{c_{13}}{c_{11}c_{33} - c_{13}^2}\right\}\nabla_1^2 P, \tag{20b}$$

$$\Lambda = \frac{h^3}{3}\left\{\frac{c_{33}(c_{11} + c_{12}) - 2c_{13}^2}{c_{11}c_{33} - c_{13}^2}\right\}\left\{\Omega'(z) + \bar{\Omega}'(\bar{z})\right\} - \frac{2h^2c_{13}(c_{11} - c_{12})p}{5(c_{11}c_{33} - c_{13}^2)} + \frac{c_{33}(c_{11} + c_{12}) - 2c_{13}^2}{c_{11}c_{33} - c_{13}^2}\nabla_1^2 P, \tag{20c}$$

$$\Gamma = \frac{16h^3(c_{11} - c_{12})}{3} \frac{\partial^2}{\partial \bar{z}^2} (\phi + i\psi), \tag{20d}$$

$$\Psi_0 = \frac{10hc_{44}}{3} \frac{\partial}{\partial \bar{z}} (w^* + 2\phi + 2i\psi), \tag{20e}$$

$$D^* = 4 \frac{\partial}{\partial \bar{z}} (\phi + i\psi). \tag{20f}$$

The function ψ satisfies the equation

$$\psi - \frac{c_{11} - c_{12}}{c_{44}} \cdot \frac{h^2}{5} \nabla_1^2 \psi = 0. \tag{20g}$$

SOME REMARKS ON THE FLAT PLATE BOUNDARY LAYER*

By J. A. LEWIS and G. F. CARRIER (*Brown University*)

1. Introduction. In a recent paper[1], the Blasius solution for the boundary layer flow past a flat plate was supplemented by an investigation of the character of the flow field near the leading edge of the plate. In that paper, the leading edge solution (of the Stokes type) was matched numerically to the Blasius solution. At first glance, therefore, it would seem desirable to find the solution of the Oseen type and either verify or improve this match. In the present paper we shall compute the exact solution of the Oseen equations associated with the flow past a semi-infinite plate and shall present arguments which verify our belief that this solution is physically not acceptable. However, we shall introduce a modification of the Oseen linearization such that the exact solution of the modified equations completes the flow pattern with a reasonable degree of accuracy.

We shall also indicate an iteration procedure from which the exact flow pattern (i.e., the exact solution of the non-linear equations) can be obtained as the limit of a rapidly converging sequence of functions. Since the actual calculation associated with each step of this iteration would be very tedious, and since the solution given by Blasius[1] and the present result give the most interesting information, we shall not complete the integrations.

Finally, we shall indicate the "modified Oseen solution" for the flow past a flat plate of finite length. Again, the interest does not seem to justify the necessary algebra so that no numerical results are obtained.

2. The Oseen linearization. The equations which govern the flow of a viscous, incompressible fluid

$$u_i \frac{\partial u_i}{\partial x_i} + \rho^{-1} \frac{\partial p}{\partial x_i} = \nu \frac{\partial^2 u_i}{\partial x_i \, \partial x_i}, \tag{1}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2}$$

*Received Nov. 5, 1948.

[1]G. F. Carrier and C. C. Lin, *On the nature of the boundary layer near the leading edge of a flat plate*, Q. Appl. Math. **6**, 63-68 (1948).

can be linearized in a manner first suggested by Oseen[2]. For our problem, the arguments leading to such a linearization might be the following. The stream function and velocity functions defining the flow must have a branch point at the leading edge of the plate (i.e., the origin) and, furthermore, the velocities vanish at this point. This implies that the third term in Eq. (1) will provide much larger contributions near the origin than will the first term. Moreover, the velocity at large distances from the plate is essentially the constant free stream velocity, u_0 . Thus, one replaces Eq. (1) by

$$u_0 \frac{\partial u_j}{\partial x_1} + \rho^{-1} \frac{\partial p}{\partial x_j} = \nu \frac{\partial^2 u_j}{\partial x_i \, \partial x_i} \tag{3}$$

and anticipates that the solution of Eq. (3) will demonstrate the proper qualitative behavior very near the plate and very far from the plate. However, if the very low speed field is separated from the "far field" by a region in which the non-linear terms predominate, we shall be faced with a phenomenon not unlike the Stokes phenomenon encountered when investigating asymptotic solutions of ordinary differential equations. In fact, we *cannot* anticipate that the solution of Eqs. (2) and (3), which holds for the far field, will be the same solution which holds in the slow flow region even though in each region the asymptotic form of the flow functions must be solutions of these equations. Nevertheless, it is profitable to find the solution of these Oseen equations and then utilize the extra information at our disposal to modify them and render the theory more applicable to our problem. Therefore, let us define $u_1 = u_0[1 + \psi_y]$, $u_2 = -u_0\psi_x$, $x = u_0x_1/\nu$, $y = u_0x_2/\nu$, where x_1 , x_2 , u_1 , u_2 are the physical coordinates and the associated velocity components. With these definitions, Eqs. (2) and (3) become

$$\Delta\left(\Delta - \frac{\partial}{\partial x}\right)\psi = 0 \tag{4}$$

where Δ is the Laplacian operator. The boundary conditions require that ψ_x , $\psi_y \to 0$ far from the plate and that $\psi_x = 0$, $\psi_y = -1$ on the plate, (i.e., on $y = 0$, $x > 0$). We note here that ψ, ψ_y , ψ_{yyy} are continuous across the plate but that ψ_{yy} will be discontinuous. Indeed, let us define $f(x)$ to be the discontinuity in ψ_{yy} across the plate and note that $f(x)$ vanishes when $x < 0$ and that f is twice the velocity gradient ($\partial u_1/\partial x_2$ in dimensionless form) along the plate. We may most readily formulate the problem by the use of Fourier transforms. In fact, we define

$$\bar{\psi}(\xi, \eta) = \int_{-\infty}^{\infty} \psi(x, y) e^{-i(\xi x + \eta y)} \, dx \, dy, \tag{5}$$

multiply Eq. (4) by $\exp[-i(\xi x + \eta y)]$, and integrate over the infinite region. Under our boundary conditions, integration by parts with respect to y leads to

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{\partial^2}{\partial x^2} - \eta^2\right)\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} - \eta^2\right)\psi(x, y) e^{-i(\xi x + \eta y)} \, dx \, dy = \int_{-\infty}^{\infty} - i\eta f(x) e^{-i\xi x} \, dx, \tag{6}$$

where $f(x)$ is defined in the foregoing. Defining $\bar{f}(\xi)$ in the obvious manner and completing the integration, we obtain

$$(\xi^2 + \eta^2)(\xi^2 + \eta^2 + i\xi)\bar{\psi} = - i\eta\bar{f}. \tag{7}$$

[2]H. Lamb, *Hydrodynamics*, Dover Publications, New York, 1945, p. 610.

At this stage, we can avoid future difficulties by observing that Eq. (7) is the limit (as $k \to 0$) of the equation

$$(\xi^2 + \eta^2 + k^2)(\eta^2 + [\xi + i][\xi - ik])\bar{\psi} = -i\eta\bar{f}. \tag{8}$$

We also note that the boundary conditions $\psi_v = -1$ on the plate may be written as the limit, as $\alpha \to 0$, of $\psi_v(x, 0) = -e^{-\alpha x}$ on $x > 0$. Therefore, we now are prepared to solve Eq. (8) for $\bar{\psi}$ and f according to the boundary conditions, with the obvious intent of carrying out the indicated limiting processes. Let us now define the function $\bar{\varphi}(\xi, y)$ to be

$$\begin{aligned} \bar{\varphi}(\xi, y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\psi}(\xi, \eta)e^{i\eta y}\, d\eta \\ &= \int_{-\infty}^{\infty} \psi(x, y)e^{-i\xi x}\, dx. \end{aligned} \tag{9}$$

The latter identity follows from the conventional inversion formula. Solving Eq. (8) for $\bar{\psi}$ and performing the integration in Eq. (9) by the use of simple contour integrals, we obtain

$$\bar{\varphi}(\xi, y) = \frac{\bar{f}[\exp\{-|y|(\xi^2 + k^2)^{1/2}\} - \exp\{-|y|(\xi + i)^{1/2}(\xi - ik)^{1/2}\}]}{2i(1 - k)(\xi - ik)} \tag{10}$$

and, in particular,

$$\bar{\varphi}_v(\xi, 0) = \frac{\bar{f}(\xi)}{2(\xi - ik)^{1/2}[(\xi + ik)^{1/2} + (\xi + i)^{1/2}]}. \tag{11}$$

However, our boundary conditions are associated with $\bar{\varphi}_v$, and it is convenient to define

$$\bar{\varphi}_v(\xi, 0) = \bar{u}_0(\xi) + \bar{u}_1(\xi)$$

where

$$\bar{u}_0(\xi) = -\int_0^{\infty} e^{-\alpha x}e^{-i\xi x}\, dx = \frac{i}{\xi - i\alpha} \tag{12}$$

and

$$\bar{u}_1(\xi) = \int_{-\infty}^{0} u(x)e^{-i\xi x}\, dx; \tag{13}$$

that is, $\bar{u}_0$ is the transform of ψ_v along the plate and $\bar{u}_1$, of ψ_v along the line $y = 0$ $x < 0$. The latter function is, of course, not known. Using these substitutions in Eq. (11)

$$\bar{u}_1(\xi) + i/(\xi - i\alpha) = \frac{\bar{f}(\xi)}{2(\xi - ik)^{1/2}[(\xi + ik)^{1/2} + (\xi + i)^{1/2}]}. \tag{14}$$

We can now determine $\bar{f}$ and $\bar{u}_1$ by the Wiener-Hopf method.[3] We note that, by definition, $\bar{u}_1(\xi)$ is an analytic function of the complex variable ξ in some upper half-plane which includes the real axis, provided we anticipate that ψ_v should decay ex-

[3] This technique has been extensively used, of late, on diffraction problems by H. Levine, J. Schwinger, A. E. Heins, and others.

ponentially in $(-x)$ on the negative x-axis. Similarly, $\bar{f}(\xi)$ is analytic in some lower half-plane including the real axis. Therefore, when we rewrite Eq. (14) in the form

$$\bar{u}_1(\xi)[(\xi + ik)^{1/2} + (\xi + i)^{1/2}] + \frac{i}{\xi - i\alpha}\{[(\xi + ik)^{1/2} + (\xi + i)^{1/2}]$$

$$- (i)^{1/2}[(\alpha + k)^{1/2} + (\alpha + 1)^{1/2}]\} = \tfrac{1}{2}\bar{f}(\xi)(\xi - ik)^{-1/2} \qquad (15)$$

$$- \frac{i}{\xi - i\alpha}(i)^{1/2}[(\alpha + k)^{1/2} + (\alpha + 1)^{1/2}],$$

the left-hand side is analytic in the upper half-plane mentioned before and the right side is analytic in the lower half-plane. Thus, by the arguments of analytic continuation, each side is equal to a constant, which is readily seen to be zero. Consequently, in the limit $k \to 0$, $\alpha \to 0$

$$\bar{f}(\xi) = 2i\left(\frac{i}{\xi}\right)^{1/2}, \qquad (16)$$

and we find by the conventional inversion formula that

$$f(x) = 2(\pi x)^{-1/2}. \qquad (17)$$

But $f(x)$ is twice the velocity gradient along the plate. Thus we obtain $.555/x^{1/2}$ for the velocity gradient, as compared with $.332/x^{1/2}$ which, of course, is the Blasius result[1]. Thus, we must conclude that the region anticipated earlier in this section exists and that the solution is not valid.

3. The modified Oseen solution. It is most convenient to introduce our proposed modification of the Oseen linearization with a minimum of *a priori* argument and, having found the desired solution, to establish justification arguments using the result to strengthen these arguments. We therefore merely suppose, for the moment, that the inertia term of Eq. (1) should be replaced, not by $u_0\partial u_i/\partial x_1$, but by $u^*\partial u_i/\partial x_1$, where u^* is some weighted average of the horizontal velocity component. It is evident that different methods of averaging would give different u^* values so we may write $u^* = cu_0$ where we expect that $0 < c < 1$. Reserving the privilege of choosing c after the equations are solved, we replace u_0 by cu_0 in Eq. (3), and Eq. (4) becomes

$$\Delta\left(\Delta - c\frac{\partial}{\partial x}\right)\psi = 0. \qquad (18)$$

It is evident that the solution can be found as before, and in fact, we now obtain

$$f(c, x) = 2(c/\pi x)^{1/2} \qquad (19)$$

where f is again the jump in ψ_{yy} across the plate.

It can also be shown after some manipulation that

$$\psi \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\varphi}(\xi, y)e^{i\xi x}\, d\xi \qquad (20)$$

can be written

$$\psi = \int_{-\infty}^{x} [(r + x)/2c\pi r^2]^{1/2}(1 - e^{c(x-r)/2})\, dx, \qquad (21)$$

and that, in particular

$$v' = -\psi_x = -[(r + x)/2c\pi r^2]^{1/2}(1 - e^{c(x-r)/2}) \tag{22}$$

$$u' = \psi_y = \int_{-\infty}^{x} \frac{\partial}{\partial y} \{[(r + x)/2c\pi r^2]^{1/2}(1 - e^{c(x-r)/2})\}\, dx \tag{23}$$

where $r^2 = x^2 + y^2$.

We now observe that the velocity gradient at the plate which is associated with the solution agrees with the Blasius result when c is chosen to be about .35. We also observe that the character of the far field flow is unaffected by changes in c provided $c > 0$. That is, the leading terms in the dimensionless velocities u' and v' are essentially independent of c for large distances from the plate. Thus the following observations are valid: the modified Oseen equation (18) is as valid as Eq. (4) for predicting the far field behavior and the Stokes (slow) flow field for any positive c. The choice $c = .35$ allows us to use a single solution of Eq. (18) to predict these asymptotic flows and also to *approximate* the flow in the intermediate region. Since any Oseen attack is intended to do no more than this, it appears that this modification constitutes an improvement (for this particular problem) over the conventional attack. It must be noted, however, that we needed to know the value of ψ_{yy} at the plate in order to choose our correction.

There are various ways in which one might formulate a criterion for choosing c when the velocity gradient at the obstacle is not known. We might, for example, demand that the average (along the negative real axis) of the neglected inertia term be made to vanish, that is, that

$$\int_{-\infty}^{0} (1 + u' - c) \frac{\partial u'}{\partial x}\, dx = 0.$$

This, in fact, yields $c = 1/2$. A second way would be to weight this average. For example, if we wished to make the inertia term have mean value zero in a wedge shaped region ahead of the plate, we could require that

$$\int_{-\infty}^{0} x(1 + u' - c) \frac{\partial u'}{\partial x}\, dx = 0.$$

Unfortunately, however, this integral does not exist, even in the Abelian sense. However, using the argument that when u' is very small the value of c does not matter, we might weight the inertia term with u' itself. That is, we might require that

$$\int_{-\infty}^{0} u'(x, 0)[1 + u' - c] \frac{\partial u'}{\partial x}\, dx = 0.$$

This produces the result $c = 1/3$ which is, of course, an excellent approximation to the value formerly found. Since this averaging makes sense physically and yields the proper result in at least one case, we suggest that, in general, the Oseen linearization associated with the viscous flow past an obstacle can be improved by using u^* instead of u_0, where u^* is found by the foregoing averaging process. However, in view of the fact that this yields $u^* = cu_0 = u_0/3$ without regard to the form of u' (and therefore without dependence on the symmetric obstacle shape), one must immediately conclude that $u^* \simeq u_0/3$ for any symmetric obstacle. The flow so obtained can then be expected to be valid in the Stokes slow flow region near the obstacle, far from the obstacle, and

in some strip along $x < 0$, $| y | < y_0$. This is certainly as large a region of validity as one could expect from such an asymptotic solution.

4. The exact solution. If we now write down the exact equation for the stream function ψ (as defined before) we obtain

$$\Delta\left(\Delta - c\frac{\partial}{\partial x}\right)\psi = F(\psi), \tag{24}$$

where $F(\psi)$ is that portion of the inertia term not contained in $c\psi_x$. We can now find a Green's function for this equation (treating F as a non-homogeneous term) and write

$$\psi = \psi_0(x, y) + \iint_{-\infty}^{\infty} G(x, x_0 ; y, y_0)F(\psi[x_0 , y_0]) \, dx_0 \, dy_0 , \tag{25}$$

where ψ_0 is the modified Oseen solution just obtained. We note that if a function ψ^*, which was very nearly the exact solution of Eq. (24), were placed in the integral term, the function ψ obtained from the integration and addition of ψ_0 would, in general, be still more accurate. This, of course, is a standard procedure. However, we have at our disposal the fact that for $|$ arc tan $(y/|\, x\,|)\,| <$ (say) $\pi/18$, the Blasius solution is very accurate, and for the remaining field the present solution seems adequate. Therefore, if one divided the integral of Eq. (25) into two parts, chose ψ^* as suggested here, and carried out the integration, he could expect a marked improvement in the result. Actually, the results already in hand seem good enough so that the amount of labor required is essentially prohibitive. Thus we shall not pursue the question further except to write down the appropriate Green's function. When ψ^* is odd in y, the integral of Eq. (25) will always obey the conditions implying $\psi = \psi_y = 0$ on $y = 0$, $x > 0$, if the Green's function is

$$G_0 = \operatorname{Re} \iint_{-\infty}^{\infty} \ln\,[(z^{1/2} - z_1^{1/2})(z^{1/2} - \bar{z}_1^{1/2})]e^{c(x_1 - x_0)/2}$$

$$\cdot\, K_0\left\{\frac{c}{2}\,[(x_1 - x_0)^2 + (y_1 - y_0)^2]^{1/2}\right\} dx_1 \, dy_1 ,$$

where $z = x + iy$; $0 < \arg z^{1/2} < \pi$; $\pi < \arg\,(\bar{z})^{1/2} < 2\pi$; and K_0 is the modified Hankel function of order zero.

5. The plate of finite length. The technique by which the foregoing problem was solved cannot be directly utilized when the plate is of finite length. However, if we let the plate extend over the interval $0 < x < A$, we may write (for the jump in ψ_{yy})

$$f(x) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} a_n x^{n-1/2} & \text{for} \quad 0 < x < A \\ 0 & \text{for} \quad x < 0, \quad \text{and} \quad x > A. \end{cases}$$

When this is done, $\psi_y(x, 0)$ in $0 < x < A$ can be written

$$\psi_y(x, 0) = \sum_n a_n' x^n \exp\,(k_n x).$$

It can also readily be shown that one may successively find the a_n' (and hence a_n) so that

$$\psi_y(x, 0) = -1 \qquad \text{on} \qquad 0 < x < A.$$

The series will fail to converge, of course, at $x = A$. However, a_0 can readily be shown to exceed 1.11 (the value obtained for the semi-infinite plate) and thus the finite plate consideration does not make the unmodified Oseen procedure more valid. In fact, one might use the modified procedure to improve the finite plate solution although, for this problem, we do not know the velocity gradient at the plate as we did for the long plate. In any event, the numerical evaluation of the velocity field will not be pursued here in any detail.

AN ITERATIVE METHOD FOR FINDING THE SMALLEST EIGENVALUE OF A MATRIX*

By JOHN B. LOHMAN (*Brown University*)

The eigenvalues and eigenvectors of a matrix A are often found by repeated premultiplication of an arbitrary column vector v by the matrix A. The products $A^n v$ converge to the eigenvector v_1 corresponding to the eigenvalue λ_1 largest in absolute magnitude; convergence is rapid if λ_1 is much larger than any other eigenvalue of A.

If one already has approximate values of the eigenvalues of A, this iterative method can be modified to give more rapid convergence. Aitken[1] suggests subtracting a scalar matrix aI (I denotes the unit matrix) from A; a is chosen so as to give the maximum separation of λ_1 . Kincaid[2] carries out the iteration using a polynomial $P(A)$ in A, in place of A itself; $P(A)$ is so chosen that the eigenvalues not immediately sought are as nearly zero as possible.

Iteration for the smallest eigenvalue of a matrix A is commonly performed using the inverse matrix A^{-1}. This note describes a modified procedure leading to more rapid convergence.

Let the eigenvalues of A be .22, .4, 1, 2, and assume we know them only to within 10%. Simple inversion of A yields A^{-1} with eigenvalues 4.5, 2.5, 1, .5; the largest eigenvalue of A^{-1} (corresponding to the smallest eigenvalue of A) is 1.8 times as large as the second-largest. Assume we believe the smallest eigenvalue of A to be .2 instead of .22; subtract the scalar matrix .2I from A, and then invert; the resulting matrix $(A - .2I)^{-1}$ has eigenvalues 50, 5, 1.25, .56. But we can still improve the separation by subtracting another scalar matrix. We choose the mean of the second largest and the smallest eigenvalues (a different choice would be appropriate if some eigenvalues were negative), for maximum separation of the largest and the second largest eigenvalues. Our final modified matrix, $(A - .2I)^{-1} - 2.78I$, has eigenvalues 47.22, 2.22, −1.53, −2.22. The ratio of largest to second largest eigenvalue has been increased from 1.8 to 21.3.

Since the modified matrix has all but one of its eigenvalues tightly clustered about zero, it is not a suitable form from which to determine these other eigenvalues.

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[1] A. C. Aitken, Proc. Roy. Soc. Edinburgh **57**, 269-270 (1937).

[2] W. M. Kincaid, Q. Appl. Math. **5**, 320-345 (1947).

AN ALTERNATE PROOF OF THE CONSTANCY OF CIRCULATION*

By WALLACE D. HAYES (*Brown University*)

In a non-viscous, barotropic fluid with only conservative body forces the circulation about any circuit that moves with the fluid is constant. The direct proof of this theorem due to Kelvin is the one customarily given but other proofs due to Cauchy and Helmholtz are also available. By the application of Stokes' theorem the statement may be changed to the constancy of the flux of vorticity through a surface bounded by a contour, both of which move with the fluid. The theorem in this latter statement is here proven directly, using vector language.

Let $\mathbf{A}$ be a vector which satisfies the equation

$$\frac{D\mathbf{A}}{Dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{A} = (\mathbf{A}\cdot\nabla)\mathbf{q} - \mathbf{A}(\nabla\cdot\mathbf{q}), \tag{1}$$

where $\mathbf{q}$ is the velocity vector and D/Dt the total time derivative.

A differential surface element $\mathbf{dS}$ satisfies the kinematic condition

$$\frac{D\mathbf{dS}}{Dt} = -(\mathbf{dS}\cdot\nabla)\mathbf{q} + \mathbf{dS}(\nabla\cdot\mathbf{q}) - \mathbf{dS}\times(\nabla\times\mathbf{q}). \tag{2}$$

This condition may be derived by considering $\mathbf{dS}$ to be the vector product $(\mathbf{a}\times\mathbf{b})$ of two differential distance elements lying in the surface. The total derivative may be expressed

$$\frac{D(\mathbf{a}\times\mathbf{b})}{Dt} = (\mathbf{a}\cdot\nabla)\mathbf{q}\times\mathbf{b} + \mathbf{a}\times(\mathbf{b}\cdot\nabla)\mathbf{q}$$

$$= -[(\mathbf{a}\times\mathbf{b})\times\nabla]\times\mathbf{q}$$

which leads to relation (2). Taking the scalar product of Eq. (1) with $\mathbf{dS}$ and of Eq. (2) with $\mathbf{A}$ and adding yields

$$\frac{D(\mathbf{A}\cdot\mathbf{dS})}{Dt} = 0, \tag{3}$$

the terms on the right cancelling because of vector identities. If the integral of this quantity be taken over a surface which moves with the fluid and its bounding contour, the result is obtained:

$$\frac{D}{Dt}\iint \mathbf{A}\cdot\mathbf{dS} = 0, \tag{4}$$

i.e., the flux of the vector through the surface is constant. It may be noted that $\nabla\cdot\mathbf{A}$ necessarily satisfies the same continuity equation as does the density.

If a vector field $\mathbf{B}$ satisfies the condition

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{q}\times(\nabla\times\mathbf{B}) + \nabla\psi, \tag{5}$$

*Received Oct. 21, 1948.

where ψ is an arbitrary scalar, its curl satisfies condition (1). The result (4) together with Stokes' theorem gives

$$\frac{D}{Dt}\oint \mathbf{B}\cdot d\mathbf{l} = 0, \tag{6}$$

i.e., that the line integral of **B** about a circuit which moves with the fluid is constant. This may also be proven directly.

The velocity vector **q** satisfies (5) as a dynamical condition and the vorticity vector $\nabla \times \mathbf{q}$ satisfies (1). Thus (4) gives the theorem for the constancy of flux of vorticity and (6), the constancy of circulation.

A GENERAL APPROXIMATION METHOD IN THE THEORY OF PLATES OF SMALL DEFLECTION

A REMARK ON MY PAPER

QUARTERLY OF APPLIED MATHEMATICS, **6**, 31-52 (1948)

BY M. Z. KRZYWOBLOCKI (*University of Illinois*)

The author wishes to apologize to Professor R. V. Southwell and Dr. A. Weinstein for not mentioning their methods in his paper. As Professor Southwell wrote in one of his papers, it is unlikely, in present circumstances, that any review of numerous methods which appeared lately in the theory of plates could be made complete and any judgement regarding the merits of different methods is premature.

So let us add just two more methods to those mentioned in the Introduction (p. 31) and on the bottom of p. 42 of the author's paper. Weinstein's variational method allows one to solve problems of plates of arbitrary shapes governed by the differential equation $\Delta\Delta w = q$, i.e. with a transverse load only (see, A. Weinstein, Mémorial des sciences mathématiques, No. 88, 1937; N. Aronszajn and A. Weinstein, Am. Jour. Math. **64**, 623-43 (1942); A. Weinstein and D. H. Rock, Quart. App. Math., **2**, 262-6 (1944); A. Weinstein and J. A. Jenkins, Trans. R. S. Canada, Sec. III, 60-67 (1946). Southwell's relaxation method, from its nature is applicable to all cases as may be seen from the examples in Southwell's "Relaxation Methods Applied to Engineering Problems" and from the following papers: L. Fox and R. V. Southwell, Trans. Roy. Soc. (A) **239**, 419-460 (1945); D. G. Christopherson, L. Fox, J. R. Green, F. S. Shaw, and R. V. Southwell, *ibid.* **239**, 461-487 (1945).

BOOK REVIEWS

Tables of Bessel functions of fractional order. Prepared by the Computation Laboratory of the National Applied Mathematics Laboratories, National Bureau of Standards, Volume I. Columbia University Press, New York, 1948. XLII + 413 pp. $7.50.

The first part (pp. 1-272) contains tables for $J_\nu(x)$ for $\nu = \pm 1/4, \pm 1/3, \pm 2/3, \pm 3/4$ and x ranging from 0 to 25, at intervals of 0.001 for small values of x(up to 0.9 for $\nu = -3/4, -2/3$, to 0.8 for $\nu = -1/3, -1/4$, to 0.6 for $\nu = 1/4, 1/3$, and to 0.5 for $\nu = 2/3, 3/4$) and at intervals of 0.01 for larger

values of x. The second part (pp. 273-383) gives the functions $A_\nu(x)$ and $B_\nu(x)$ defined by $J_\nu(x) = A_\nu(x) \cos(x - \pi\nu/2 - \pi/4) - B_\nu(x) \sin(x - \pi\nu/2 - \pi/4)$ for the same values of ν and for $x = [25\ (.1)\ 50\ (1)\ 500\ (10)\ 5{,}000\ (100)\ 10{,}000\ (200)\ 30{,}000]$. All functions are tabulated to ten decimal places.

A number of auxiliary tables facilitate interpolation and give the zeros of $J_\nu(x)$.

W. PRAGER

Tables of the Bessel functions of the first kind of orders zero and one. By the Staff of the Computation Laboratory (The Annals of the Computation Laboratory of Harvard University—Volume III). Harvard University Press, Cambridge, 1947. $10.00.

Following a brief discussion of the Bessel functions, the computations, the interpolations, and a list of the Bessel coefficients, the Bessel functions $J_0(x)$ and $J_1(x)$ are tabulated to eighteen digits. The argument varies in steps of .001 for $0 \leq x \leq 25$, and in steps of .01 for $25 < x < 100$.

G. F. CARRIER

Tables of the Bessel functions of the first kind of orders two and three. By the Staff of the Computation Laboratory (The Annals of the Computation Laboratory of Harvard University—Volume IV). Harvard University Press, Cambridge, 1947. $10.00.

The Bessel functions $J_2(x)$ and $J_3(x)$ are tabulated to eighteen digits. The argument varies in steps of .001 in the range $0 \leq x \leq 25$, and in steps of .01 in the range $25 < x < 100$.

G. F. CARRIER

Tables of the Bessel functions of the first kind of orders four, five and six. By the Staff of the Computation Laboratory (The Annals of the Computation Laboratory of Harvard University—Volume V). Harvard University Press, Cambridge, 1947. $10.00.

The Bessel functions $J_4(x)$, $J_5(x)$ and $J_6(x)$ are tabulated to ten digits. The argument varies in steps of .001 in the range $0 \leq x \leq 25$, and in steps of .01 in the range $25 < x < 100$.

G. F. CARRIER

Tables of the Bessel functions of the first kind of orders seven, eight and nine. By the Staff of the Computation Laboratory (The Annals of the Computation Laboratory of Harvard University—Volume VI). Harvard University Press, Cambridge, 1947. $10.00.

The Bessel functions $J_7(x)$, $J_8(x)$ and $J_9(x)$ are tabulated to ten digits. The argument varies in steps of .001 in the range $0 \leq x \leq 25$, and in steps of .01 in the range $25 < x < 100$.

G. F. CARRIER

Tables of the Bessel functions of the first kind of orders ten, eleven and twelve. By the Staff of the Computation Laboratory (The Annals of the Computation Laboratory of Harvard University—Volume VII). Harvard University Press, Cambridge, 1947. $10.00.

The Bessel functions $J_{10}(x)$, $J_{11}(x)$ and $J_{12}(x)$ are tabulated to ten digits. The argument varies in steps of .001 in the range $0 \leq x \leq 25$, and in steps of .01 in the range $25 < x < 100$.

G. F. CARRIER

Tables of the Bessel functions of the first kind of orders thirteen, fourteen and fifteen. By the Staff of the Computation Laboratory (The Annals of the Computation Laboratory of Harvard University—Volume VIII). Harvard University Press, Cambridge, 1947. $10.00.

The Bessel functions $J_{13}(x)$, $J_{14}(x)$ and $J_{15}(x)$ are tabulated to ten digits. The argument varies in steps of .001 in the range $0 \leq x \leq 25$, and in steps of .01 in the range $25 < x < 100$.

G. F. CARRIER

Tables of the Bessel functions of the first kind of orders sixteen through twenty-seven. By the Staff of the Computation Laboratory (The Annals of the Computation Laboratory of Harvard University—Volume IX). Harvard University Press, Cambridge, 1948. $10.00.

The Bessel functions $J_{16}(x)$, $J_{17}(x)$, ... , $J_{27}(x)$ are tabulated to ten digits. The argument varies in steps of .01 for $0 \leq x < 100$.

G. F. CARRIER

Tables of the Bessel functions of the first kind of orders twenty-eight through thirty-nine. By the Staff of the Computation Laboratory (The Annals of the Computation Laboratory of Harvard University—Volume X). Harvard University Press, Cambridge, 1948. $10.00.

The Bessel functions $J_{28}(x)$, $J_{29}(x)$, ... , $J_{39}(x)$ are tabulated to ten digits. The argument varies in steps of .01 for $0 < x < 100$.

G. F. CARRIER

Eleven and fifteen-place tables of Bessel functions of the first kind, to all significant orders. By Enzo Cambi. New York, Dover Publications, 1948. vi + 154 pp. $3.95.

This new work is in two main parts. Table I (117 pages) gives $J_n(x)$ for $x = 0\,(.01)10.5$ to 11 decimals for all integers n for which $| J_n(x) | \leq 5 \cdot 10^{-12}$, that is for $| n | \leq 30$. Table II (26 pages) is a 15-place table of $J_n(x)$ for $x = 0(.001).5$ extending over all integers n for which $| J_n(x) | \leq 10^{-15}$, that is for $| n | \leq 11$.

There are two auxiliary tables of coefficients to 12 decimals. The first of these gives the coefficients

$$c_\nu = c_\nu(k, n) = J_n^{(\nu)}(k)/\nu!$$

in the Taylor series

$$J_n(k + h) = \sum_{\nu=0} c_\nu h^\nu \qquad (n = 2(2)16,\ k = 2(1)6,\ \nu \leq 10, 11).$$

The second auxiliary table is of A_m in

$$J_n(k + h) = \sum_{m=0}^{q} A_m J_m(h) \qquad (n = 2(2)30,\ k = 7(1)10).$$

The coefficients A_m of this Neumann series are given by the formulae

$$A_0 = A_0(k, n) = J_n(k)$$

$$A_m = A_m(k, n) = J_{n-m}(k) + (-1)^m J_{n+m}(k) \qquad (m > 0)$$

which follow at once from the well known addition theorem for $J_n(x)$.

These auxiliary tables were used by the author to build up the values in Table I for n even and ≥ 2. (J_0 and J_1 were taken from Meissel's well known table). The values for n odd were then computed from the recurrence formula

$$J_{2r+1}(x) = x(4r + 2)^{-1}[J_{2r}(x) + J_{2r-2}(x)].$$

An unusual feature of Tables I and II is a "check" column showing for each value of x the errors made in substituting the tabulated values into the identities

$$1 = \sum_{n=-\infty}^{\infty} J_{2n}(x),$$

$$\sin x = 2 \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(x),$$

$$x/2 = \sum_{n=0}^{\infty} (2n + 1) J_{2n+1}(x).$$

These errors are all satisfactorily small. Unfortunately these checks were not applied to the proof sheets. Otherwise the following typographical error (noted by the author) might have been detected in time for correction: p. 47, $J_5(6.93)$, for .35385 88467 2 read .35385 88367 2.

Another identity, used by the reviewer in spot checking the table, is the Kapteyn series

$$(1 - z)^{-1} = 1 + 2 \sum_{n=1}^{\infty} J_n(nz).$$

There is no discussion of interpolation nor are any differences tabulated.

This compact and relatively inexpensive table of $J_n(x)$ will prove very useful to almost all workers with these fundamental functions. As explained in the introduction, the table emphasizes the Bessel functions of the higher orders n. Its choice of the variables n, x will make it especially useful in dealing with Neumann series

$$\sum_{n=-\infty}^{\infty} a_n J_n(x).$$

However for some applications, such as certain Fourier-Bessel series, the limitation $x \leq 10.5$ may prove too restrictive. In fact the table gives very few waves of the wavy surface $z = J_y(x)$. Thus, within the range of tabulation, J_0 and J_1 have only 3 zeros, J_2 and J_3 have 2 and J_4, J_5, J_6 have one each. J_7 and J_8 have 1 maximum each while J_9 and J_{10} only one point of inflexion each. All other functions are monotone with monotone derivatives.

The offset printing from typescript is very clear and well set out.

D. H. Lehmer

Introduction to the differential equations of physics. By L. Hopf. Translated by Walter Nef. Dover Publications, Inc., New York, 1948. v + 154 pp. $1.95.

This book is an introduction to the differential equations of mathematical physics. The exposition is clear and concise, and one might like the treatment to go further than it does. The text should be very useful to seniors and beginning graduate students in physics and chemistry.

The contents of the book are divided into seven chapters. The first four chapters are devoted to discussion of the ordinary differential equations of the mechanics of particles and the simpler partial differential equations of physics as applied to potential theory, heat conduction, potential flow, and elec-

tromagnetic theory. Chapter V is devoted to solutions of the partial differential equations by eigenfunctions with a discussion of the product method, Fourier series methods, expansions, regular and asymptotic, the Fourier integral. Chapter VI discusses the solution of partial differential equations by change of variables methods; this section includes a brief discussion of conformal transformations. Chapter VII is a discussion of the use of singularities in obtaining solutions of equations. The chapter contains some discussion of the use of Green's Theorem, and the transition from differential equations to integral equations.

R. TRUELL

Introduction to applied mathematics. By Francis D. Murnaghan. John Wiley & Sons, Inc., New York and Chapman & Hall, Ltd., London, 1948. ix + 389 pp. $5.00.

The text begins with a development of the algebra and calculus of vector analysis based on a Cartesian reference system. The material is extended to include vectors in an n-dimensional space and then Fourier series are introduced from the function space point of view. In chapter four the vector calculus is carried over into the curvilinear coordinate system. A chapter then follows on the conventional method for solving the more elementary problems involving the Laplace equation. Following a chapter on Legendre and Bessel functions, concerned largely with power series developments, zeros and orthogonality, is a discussion of eigenfunctions and eigenfunction expansions associated with boundary value problems. The final chapters deal with the Fredholm equation, the calculus of variations, and the operational calculus.

On the whole, the material is sufficiently elementary and presented in such detail as to be of interest as a text for a senior or introductory graduate course in applied mathematics.

G. F. CARRIER

Introduction to complex variables and applications. By Ruel V. Churchill. McGraw-Hill Book Company, Inc., New York, Toronto and London, 1948. vi + 216 pp. $3.50.

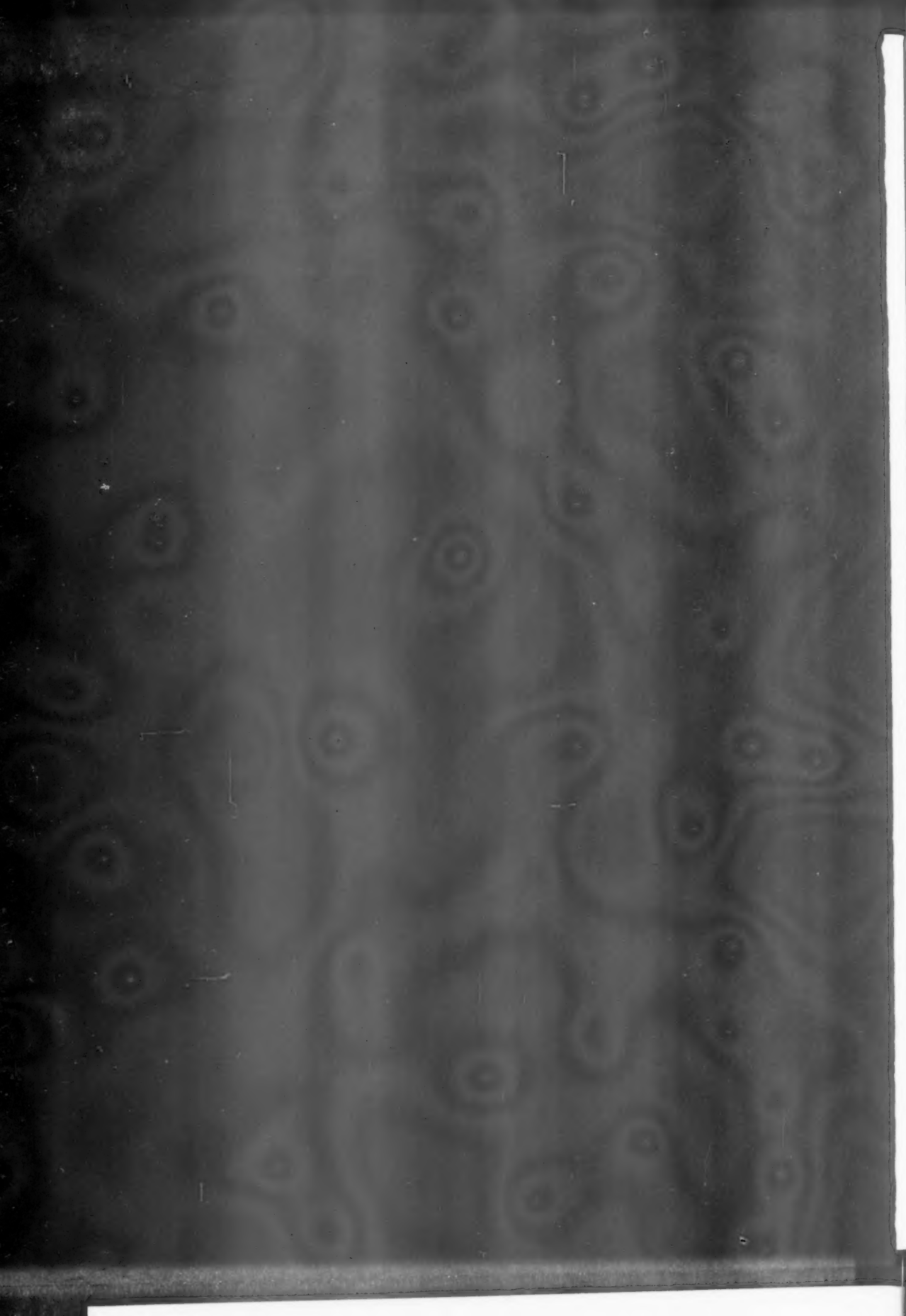
The algebra of complex numbers and the introduction of analytic functions are followed by a discussion of the elementary functions from both the analytic and geometric point of view. The integration of complex functions is developed and power series are discussed with a very brief section on Laurent series. The residue theorem is discussed with emphasis (via examples) on the evaluation of definite integrals. Conformal mapping is discussed and applied to the conventional heat transfer, electrostatic, and fluid mechanics problems. The Schwarz-Christoffel transformation is treated in detail and a brief section on analytic continuation and Riemann surfaces complete the text. The appendix contains a short "dictionary" of simple conformal mappings. The book should be of interest to those concerned with the use of function theory in engineering problems and similar applications.

G. F. CARRIER

Principles of jet propulsion and gas turbines. By M. J. Zucrow. John Wiley & Sons, Inc., New York and Chapman & Hall, Ltd., London. xiv + 563 pp. $6.50.

A discussion of the properties of gases and the fundamental laws which they obey is accompanied by detailed material on airplane performance, propellers, gas turbines, compressors and jet engines. High temperature alloys are also discussed. A large proportion of the text is devoted to related topics rather than to the principles of jet propulsion. The emphasis is on the presentation of engineering information and the text is perhaps most appropriate for an engineering course at the senior or early graduate level.

G. F. CARRIER



SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

Mathematical Work: Only very simple symbols and formulas should be typewritten. All others should be carefully written by hand in ink. Ample space for marking should be allowed above and below all equations. Greek letters used in formulas should be designated by name in the margin.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter *O*, between the numeral one (1), the letter *l* and the prime (′), between alpha and *a*, kappa and *k*, mu and *u*, nu and *v*, eta and *n*.

All subscripts and exponents should be clearly marked, and dots, bars, tildes, etc. over letters should be avoided.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\sqrt{}$.

Complicated exponents and subscripts should be avoided. Any complicated expression that reoccurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp\,[(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos\,(\pi x/2b)}{\cos\,(\pi a/2b)} \text{ is preferable to } \frac{\cos\dfrac{\pi x}{2b}}{\cos\dfrac{\pi a}{2b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u\, du \text{ is preferable to } \int \frac{\sin u}{u}\, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t\,(a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

For further suggestions concerning cuts, bibliography, footnotes, and abbreviations please see the inside back cover of the January issue of 1949.

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